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# Quantum motion of particles in random dynamic fields and quantum dissipation: Schrödinger equation with Gaussian fluctuating potentials

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**Abstract.** The Schrödinger equation with Gaussian dynamic fluctuating potentials is considered. This model gives a convenient example of an open quantum system coupled with a given external environment. The evolution equations for averaged correlators are derived and it is shown that a system coupled to a thermoequilibrium environment tends asymptotically with time to a thermodynamic density matrix. At high temperatures the quantum evolution equation for the density matrix can be approximated by the quasiclassical Fokker-Planck equation. The generalisation for multiparticle systems is also given. The results are illustrated by the particular examples of coupling with phonons and of two-level systems. Finally, it is shown how the problem can be reformulated in terms of continuous stochastic evolution in the space of  $\psi$  functions and a corresponding functional formalism based on the continuous Fokker-Planck equation in the space of  $\psi$  functions is presented.

## 1. Introduction

A variety of physical situations can be considered within the framework of the following simplified treatment: a small subsystem exists separately within a large system and the latter is treated as an environment with fixed given properties, while the influence of the small subsystem on the environment is neglected. Examples of this situation are plentiful and particular realisations may be illustrated by adsorbed atoms on a surface interacting with a bulk of a pattern, by electrons trapped in potential wells in condensed media, by atomic spectra of ions in turbulent plasma, etc. In its general form the problem is formulated as the description of an open quantum system in a given environment. Because a large system has many degrees of freedom, the interaction of large and small systems can be treated statistically.

The most advanced studies of this problem are based on modelling of the medium in the form of a bath of oscillators and subsequent description of the system either in terms of statistical mechanics or with the use of the Feynman influence functional method (Feynman and Vernon 1963, Feynman and Hibbs 1965, Ford *et al* 1965, Caldeira and Leggett 1983a, b, 1985, Hänggi and Mojtabai 1982, Schmid 1982, Grote and Hynes 1982, Mel'nikov and Meshkov 1983, Cortes *et al* 1985, Riseborough *et al* 1985). The major limitation of this approach consists in formulating all results in terms of various spectral oscillator densities, while direct correspondence of these variables with observable quantities must be established additionally.

Other theories are based either on semiphenomenological generalisations of the Schrödinger equation (Kostin 1972, Yasue 1978, Hasse 1979, Diosi 1986) or on the

quantisation of frictional force (Dekker 1981 and references therein, Ghosh and Hasse 1981, Nemes and de Toledo Piza 1983). Despite the many interesting results obtained within such an approach, its physical status still remains mainly heuristic.

Our investigation is based on a model of the Schrödinger equation with Gaussian dynamic fluctuating potentials. The white-noise version of this model has been considered previously by Jayannavar and Kumar (1982). We shall show, however, that finite-time correlations play an important role, especially in thermodynamic problems. As all results obtained using the Schrödinger equation can be reformulated in path integral language (Feynman and Hibbs 1965), this approach has many features in common with the Caldeira and Leggett (1983a, b) theory. Though both models are mathematically equivalent, the direct use of the Schrödinger equation is simpler in many aspects. Additional attractive features are the use of a purely quantum description and well defined physical quantities from the very beginning. This permits us to apply such general results as the quantum fluctuation-dissipation theorem or to relate the field correlators with the corresponding Green functions which are calculated straightforwardly (see, e.g., Abrikosov *et al* 1963).

The Schrödinger equation with Gaussian dynamic fluctuating potentials gives the natural quantum generalisation of the classical Langevin scheme which can be formulated in terms of the following equations:

$$m \frac{d\mathbf{v}}{dt} = -\frac{\partial U(\mathbf{r})}{\partial \mathbf{r}} - \gamma \mathbf{v} + \mathbf{f}(\mathbf{r}, t) \quad (1.1)$$

where  $U(\mathbf{r})$  is the regular static potential and  $\mathbf{f}(\mathbf{r}, t)$  is the so-called random force determined by correlators:

$$\langle \mathbf{f}(\mathbf{r}, t) \rangle = 0 \quad (1.2)$$

$$\langle f_j(\mathbf{r}, t) f_k(\mathbf{r}, t') \rangle = \Gamma_{jk} \delta(t - t'). \quad (1.3)$$

Here  $\Gamma_{jk}$  is a symmetric constant tensor. In the case of thermoequilibrium fluctuations, its structure is determined by the classical fluctuation-dissipation theorem. The white-noise approximation  $\delta(t - t')$  leads, however, to negligence of quantum discrete eigenfrequency properties in view of the uncertainty principle for energy and time,

$$\Delta E \tau_{\text{corr}} \geq \hbar \quad (1.4)$$

where  $\tau_{\text{corr}}$  is the characteristic correlation time of a noise and the energy  $\Delta E$  corresponds to transitions between discrete energy levels which determine the stochastic evolution of a system. Thus, stochastic dynamics corresponding to (1.2) and (1.3) will really be quasiclassical even for the Schrödinger equation and finite time correlations play an important role in the quantum description (see also §§ 4 and 6).

The layout of the paper is as follows. In § 2 we derive the averaged evolution equations for various correlators and then the structure of the equations and their physical meaning is illustrated in § 3 with the particular example of coupling with phonons. In § 4 we shall prove the general theorem on the asymptotic evolution to the thermodynamic density matrix of any system coupled with thermoequilibrium fluctuations of potentials. The particular example of such an evolution is considered in § 5 with the use of a two-level system. The quasiclassical Fokker-Planck equation is derived in § 6. The generalisation to multiparticle systems is given in § 7, and in § 8 we show how the problem can be reformulated in terms of stochastic evolution in the space of  $\psi$  functions. The discussion is based on the corresponding continuous

Fokker-Planck equation in the space of  $\psi$  functions. The necessary comments on the structure of the continuous Fokker-Planck equation are presented in appendix 1, while appendix 2 is devoted to one special example.

## 2. Evolution equations for averaged correlators

### 2.1. Evolution of the averaged $\psi$ function

The significance of the problem discussed in this section is twofold. First, it is interesting as a rather general and convenient example of an open quantum system coupled to a thermostat, but we should note that fluctuations may not be necessarily at thermoequilibrium and may correspond also to artificial sources of stochastic radiation (see, e.g., Akhmanov *et al* 1981).

We begin with the simplest one-particle Schrödinger equation of the form

$$\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial \mathbf{r}^2} + U(\mathbf{r})\psi + V(\mathbf{r}, t)\psi \\ &\equiv \hat{H}(\mathbf{r})\psi + V(\mathbf{r}, t)\psi. \end{aligned} \tag{2.1}$$

Here  $U(\mathbf{r})$  is a regular static potential (corresponding, e.g., to the potential well of a trap) and  $V(\mathbf{r}, t)$  is a Gaussian dynamic random potential. The potential  $V(\mathbf{r}, t)$  can be characterised in several equivalent ways. The most usual is to characterise it by two correlators:

$$\langle V(\mathbf{r}, t) \rangle = 0 \tag{2.2}$$

$$\langle V(\mathbf{r}, t) V(\mathbf{r}', t') \rangle = \Gamma(\mathbf{r} - \mathbf{r}', t - t') \tag{2.3}$$

where, as in the Gaussian case, the correlators of the higher orders are expressed through various sums of products of pair correlators. The potential  $V(\mathbf{r}, t)$  is assumed to be classical, commuting at various moments of the time function (i.e. the  $c$  number; the quantum generalisation will be given below), and for simplicity correlations are taken to be stationary and spatially homogeneous ( $\Gamma$  in equation (2.3) depends only on  $(\mathbf{r} - \mathbf{r}')$  and  $(t - t')$ ). The same correlators can be obtained with the use of the characteristic functional

$$\begin{aligned} \Phi^{(V)}\{\lambda(\mathbf{r}, \tau); t\} &= \left\langle \exp\left(i \int_0^t d\tau \int d^d \mathbf{r} \lambda(\mathbf{r}, \tau) V(\mathbf{r}, \tau)\right) \right\rangle \\ &= \exp\left(-\frac{1}{2} \int_0^t d\tau \int_0^t d\tau' \int d^d \mathbf{r} \int d^d \mathbf{r}' \lambda(\mathbf{r}, \tau) \Gamma(\mathbf{r} - \mathbf{r}', \tau - \tau') \lambda(\mathbf{r}', \tau')\right) \end{aligned} \tag{2.4}$$

by corresponding functional differentiation with respect to  $\lambda(\mathbf{r}, \tau)$  and then tending the arguments  $\lambda(\mathbf{r}, \tau)$  to zero, i.e.

$$\langle V(\mathbf{r}_1, t_1) V(\mathbf{r}_2, t_2) \rangle = \frac{\delta^2 \Phi^{(V)}\{\lambda(\mathbf{r}, \tau); t\}}{i^2 \delta \lambda(\mathbf{r}_1, t_1) \delta \lambda(\mathbf{r}_2, t_2)} \Big|_{\{\lambda(\mathbf{r}, \tau) = 0\}} \tag{2.5}$$

etc. The problem is: how do potential fluctuations influence the evolution of the  $\psi$  function?

It is convenient to transform equation (2.1) to an interaction representation (details of the mathematical technique used below can be found in Bogolyubov and Shirkov (1959), Abrikosov *et al* (1963) and Itzykson and Zuber (1980)):

$$\psi(\mathbf{r}, t) = \exp(-i\hat{H}(\mathbf{r})t/\hbar)\tilde{\psi}(\mathbf{r}, t) \quad (2.6)$$

$$i\hbar \frac{\partial \tilde{\psi}(\mathbf{r}, t)}{\partial t} = \hat{V}^{(\text{int})}(\mathbf{r}, t)\tilde{\psi}(\mathbf{r}, t) \quad (2.7)$$

where

$$\hat{V}^{(\text{int})}(\mathbf{r}, t) = \exp(i\hat{H}t/\hbar)V(\mathbf{r}, t)\exp(-i\hat{H}t/\hbar). \quad (2.8)$$

The formal solution of equation (2.7) with given initial conditions has the form

$$\tilde{\psi}(\mathbf{r}, t) = \hat{T} \exp\left(-i \int_0^t d\tau \hat{V}^{(\text{int})}(\mathbf{r}, \tau)\right)\tilde{\psi}^{(0)}(\mathbf{r}) \equiv \hat{U}(\mathbf{r}, t)\tilde{\psi}^{(0)}(\mathbf{r}). \quad (2.9)$$

Here  $\hat{T}$  is a chronologisation operator. The need for it is dictated by the non-commutivity of the operators  $\hat{V}^{(\text{int})}(\mathbf{r}, \tau)$  at different moments of time. The evolution operator  $\hat{U}(\mathbf{r}, t)$  is equal by definition to

$$\begin{aligned} \hat{U}(\mathbf{r}, t) = & 1 - \frac{i}{\hbar} \int_0^t d\tau_1 \hat{V}^{(\text{int})}(\mathbf{r}, \tau_1) \\ & + \frac{1}{2!} \left(-\frac{i}{\hbar}\right)^2 \hat{T} \int_0^t d\tau_2 \hat{V}^{(\text{int})}(\mathbf{r}, \tau_2) \int_0^{\tau_2} d\tau_1 \hat{V}^{(\text{int})}(\mathbf{r}, \tau_1) + \dots \end{aligned} \quad (2.10)$$

where the operator  $\hat{T}$  orders various products of  $\hat{V}^{(\text{int})}(\mathbf{r}, t)$  with an increase in time, i.e.

$$\hat{T}\hat{V}^{(\text{int})}(\mathbf{r}, \tau_1)\hat{V}^{(\text{int})}(\mathbf{r}, \tau_2) = \begin{cases} \hat{V}^{(\text{int})}(\mathbf{r}, \tau_1)\hat{V}^{(\text{int})}(\mathbf{r}, \tau_2) & \tau_1 > \tau_2 \\ \hat{V}^{(\text{int})}(\mathbf{r}, \tau_2)\hat{V}^{(\text{int})}(\mathbf{r}, \tau_1) & \tau_2 > \tau_1 \end{cases} \quad (2.11)$$

and the time ordering in terms of higher order is defined analogously. Let us assume that, at the initial moment  $t=0$ , the state  $\tilde{\psi}^{(0)}(\mathbf{r})$  does not depend on the random potential  $V(\mathbf{r}, t)$  (this means equivalently that all exact information about the system is known at  $t=0$ ). As can be seen from equation (2.9), the problem is then reduced to averaging the evolution operator  $\hat{U}(\mathbf{r}, t)$  on the ensemble of realisations of  $V(\mathbf{r}, t)$ . This can be performed with the use of Wick's theorem. All odd terms will be equal to zero after averaging while, for Gaussian statistics, all even terms are expressed by the various products of pair correlators, i.e.

$$\begin{aligned} & \langle \hat{T}\hat{V}(1)\hat{V}(2)\hat{V}(3)\hat{V}(4) \rangle \\ & = \langle \hat{T}\hat{V}(1)\hat{V}(2) \rangle \langle \hat{T}\hat{V}(3)\hat{V}(4) \rangle + \langle \hat{T}\hat{V}(1)\hat{V}(3) \rangle \langle \hat{T}\hat{V}(2)\hat{V}(4) \rangle \\ & \quad + \langle \hat{T}\hat{V}(1)\hat{V}(4) \rangle \langle \hat{T}\hat{V}(2)\hat{V}(3) \rangle \end{aligned} \quad (2.12)$$

etc. It is important that operators commute under  $\hat{T}$  ordering. Then after averaging the term with  $2n$  operators  $\hat{V}^{(\text{int})}(\mathbf{r}, t)$  gives  $(2n-1)!!$  (where  $(2n-1)!! = (2n-1) \times (2n-3) \dots 1$ ) identical additives. Using the equality

$$\frac{(2n-1)!!}{(2n)!} = \frac{1}{2^n n!} \quad (2.13)$$

the various terms can be easily summed and the averaged evolution operator can be rewritten finally in the form

$$\begin{aligned} \langle \hat{U}(\mathbf{r}, t) \rangle &= 1 + \frac{1}{2} \left( -\frac{i}{\hbar} \right)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \langle \hat{T} \hat{V}^{(int)}(\mathbf{r}, \tau_1) \hat{V}^{(int)}(\mathbf{r}, \tau_2) \rangle \\ &\quad + \frac{1}{4 \times 2!} \left( -\frac{i}{\hbar} \right)^4 \left( \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \langle \hat{T} \hat{V}^{(int)}(\mathbf{r}, \tau_1) \hat{V}^{(int)}(\mathbf{r}, \tau_2) \rangle \right)^2 + \dots \\ &= \exp \left( -\frac{1}{2\hbar^2} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \langle \hat{T} \hat{V}^{(int)}(\mathbf{r}, \tau_1) \hat{V}^{(int)}(\mathbf{r}, \tau_2) \rangle \right). \end{aligned} \tag{2.14}$$

This gives the corresponding evolution equation for  $\langle \tilde{\psi}(\mathbf{r}, t) \rangle$ :

$$\frac{\partial \langle \tilde{\psi}(\mathbf{r}, t) \rangle}{\partial t} = -\frac{1}{\hbar^2} \int_0^t d\tau \langle \hat{V}^{(int)}(\mathbf{r}, t) \hat{V}^{(int)}(\mathbf{r}, \tau) \rangle \langle \hat{\psi}(\mathbf{r}, t) \rangle \tag{2.15}$$

or equivalently (see equations (2.6)-(2.8))

$$\begin{aligned} \frac{\partial \langle \psi(\mathbf{r}, t) \rangle}{\partial t} &= -\frac{i}{\hbar} \hat{H}(\mathbf{r}) \langle \psi(\mathbf{r}, t) \rangle \\ &\quad - \frac{1}{\hbar^2} \int_0^t d\tau \langle V(\mathbf{r}, t) \exp(-i\hat{H}(\mathbf{r})(t-\tau)/\hbar) V(\mathbf{r}, \tau) \rangle \\ &\quad \times \exp(i\hat{H}(\mathbf{r})(t-\tau)/\hbar) \langle \psi(\mathbf{r}, t) \rangle. \end{aligned} \tag{2.16}$$

Equation (2.16) can also be derived using more traditional statistical methods. There is an important theorem for Gaussian correlations (Furutsu 1963, Novikov 1964, Donsker 1964) which states

$$\begin{aligned} \langle V(\mathbf{r}_1, t_1) R\{V(\mathbf{r}', \tau')\} \rangle \\ = \int_0^{t_1} d\tau \int d^d \mathbf{r} \langle V(\mathbf{r}_1, t_1) V(\mathbf{r}, \tau) \rangle \left\langle \frac{\delta R\{V(\mathbf{r}', \tau')\}}{\delta V(\mathbf{r}, \tau)} \right\rangle \end{aligned} \tag{2.17}$$

for  $t_1 \leq t$ . Here  $R\{V(\mathbf{r}, \tau)\}$  is a functional depending on  $V(\mathbf{r}, \tau)$  either explicitly or implicitly. Applying this theorem to equation (2.7) one obtains

$$i\hbar \frac{\partial \langle \tilde{\psi}(\mathbf{r}, t) \rangle}{\partial t} = \int_0^t d\tau \int d^d \mathbf{r}' \langle \hat{V}^{(int)}(\mathbf{r}, t) \hat{V}^{(int)}(\mathbf{r}', \tau) \rangle \left\langle \frac{\delta \tilde{\psi}(\mathbf{r}, t)}{\delta \hat{V}^{(int)}(\mathbf{r}', \tau)} \right\rangle. \tag{2.18}$$

On the other hand, one obtains from equation (2.9)

$$\begin{aligned} \frac{\delta \tilde{\psi}(\mathbf{r}, t)}{\delta \hat{V}^{(int)}(\mathbf{r}', \tau)} &= \frac{\delta}{\delta \hat{V}^{(int)}(\mathbf{r}', \tau)} (\hat{U}(\mathbf{r}; t, \tau + \varepsilon) \hat{U}(\mathbf{r}; \tau + \varepsilon, 0)) \tilde{\psi}^{(0)}(\mathbf{r}) \\ &= \hat{U}(\mathbf{r}; t, \tau) \frac{\delta \hat{U}(\mathbf{r}; \tau, 0)}{\delta \hat{V}^{(int)}(\mathbf{r}', \tau)} \tilde{\psi}^{(0)}(\mathbf{r}) \\ &= \hat{U}(\mathbf{r}; t, \tau) \frac{\delta \tilde{\psi}(\mathbf{r}, \tau)}{\delta \hat{V}^{(int)}(\mathbf{r}', \tau)} = -\frac{i}{\hbar} \hat{U}(\mathbf{r}; t, \tau) \delta(\mathbf{r} - \mathbf{r}') \tilde{\psi}(\mathbf{r}, \tau) \\ &= -\frac{i}{\hbar} \hat{U}(\mathbf{r}; t, \tau) \delta(\mathbf{r} - \mathbf{r}') \hat{U}^{-1}(\mathbf{r}; t, \tau) \tilde{\psi}(\mathbf{r}, t). \end{aligned} \tag{2.19}$$

Here  $\varepsilon$  is an infinitesimally small positive constant and  $\hat{U}^{-1}(\mathbf{r}; t, \tau)$  is the operator reciprocal to  $\hat{U}(\mathbf{r}; t, \tau)$ . We have also used the multiplicative property of the evolution operator:

$$\hat{U}(\mathbf{r}; t, t_0) = \hat{U}(\mathbf{r}; t, t_1) \hat{U}(\mathbf{r}; t_1, t_0) \quad (2.20)$$

for any moment  $t > t_1 > t_0$  and the causality principle

$$\frac{\delta \hat{U}(\mathbf{r}; t, 0)}{\delta \hat{V}^{(\text{int})}(\mathbf{r}', \tau)} = 0 \quad (2.21)$$

if  $\tau > t$ . Substitution of (2.19) into (2.18) gives

$$\begin{aligned} \frac{\partial \langle \tilde{\psi}(\mathbf{r}, t) \rangle}{\partial t} &= -\frac{1}{\hbar^2} \int_0^t d\tau \int d^d \mathbf{r}' \langle \hat{V}^{(\text{int})}(\mathbf{r}, t) \hat{V}^{(\text{int})}(\mathbf{r}, \tau) \rangle \\ &\quad \times \langle \hat{U}(\mathbf{r}; t, \tau) \delta(\mathbf{r} - \mathbf{r}') \hat{U}^{-1}(\mathbf{r}; t, \tau) \tilde{\psi}(\mathbf{r}, t) \rangle \\ &= -\frac{1}{\hbar^2} \int_0^t d\tau \langle \hat{V}^{(\text{int})}(\mathbf{r}, t) \hat{V}^{(\text{int})}(\mathbf{r}, \tau) \rangle \langle \tilde{\psi}(\mathbf{r}, t) \rangle \end{aligned} \quad (2.22)$$

which coincides with (2.15).

It is useful to analyse this result with a more trivial example. Let us consider the 'zero-dimensional' Schrödinger equation

$$i\hbar \frac{\partial \psi(t)}{\partial t} = V(t)\psi(t). \quad (2.23)$$

This gives immediately

$$\psi(t) = \exp\left(-i \int_0^t d\tau V(\tau)/\hbar\right) \psi^{(0)} \quad (2.24)$$

$$\frac{\delta \psi(t)}{\delta V(\tau)} = -\frac{i}{\hbar} \psi(t) \quad (0 \leq \tau \leq t) \quad (2.25)$$

$$\frac{\partial \langle \psi(t) \rangle}{\partial t} = -\frac{1}{\hbar^2} \int_0^t d\tau \langle V(t) V(\tau) \rangle \langle \psi(t) \rangle. \quad (2.26)$$

The action of any elementary random constituent  $V_0 \delta(t - t_0)$  leads to a corresponding jump in the phase  $\exp(-i V_0 / \hbar)$  ( $t > t_0$ ). We see that various jumps for  $t_0 < t$  are piled up to the time  $t$ . It is important that they do not smear with time. This leads to phase shifts that are non-local in time existing at all moments  $t > t_0$  and it explains the meaning of equations (2.25) and (2.26). In the multidimensional case the phase shifts local in space and time will be partially smeared due to the action of  $\hat{H}(\mathbf{r})$  and, in fact, the locality of evolution in time will be partially lost in view of the action of the operator  $\exp(i\hat{H}(\mathbf{r})(t - \tau)/\hbar)$  on  $\langle \psi(\mathbf{r}, t) \rangle$  (see equation (2.16)), but the situation remains generally unchanged.

## 2.2. Stochastic evolution of the density matrix

The other important quantity is the density matrix  $\langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle$ . Defining

$$\psi^*(\mathbf{r}, t) = \exp(iH(\mathbf{r})t/\hbar) \tilde{\psi}^*(\mathbf{r}, t) \quad (2.27)$$

the unaveraged evolution equation can be written in the form

$$i\hbar \frac{\partial}{\partial t} (\tilde{\psi}^*(\mathbf{r}_2, t) \tilde{\psi}(\mathbf{r}_1, t)) = (\hat{V}^{(int)}(\mathbf{r}_1, t) - \hat{V}^{*(int)}(\mathbf{r}_2, t)) (\tilde{\psi}^*(\mathbf{r}_2, t) \tilde{\psi}(\mathbf{r}_1, t)) \tag{2.28}$$

where  $\hat{V}^{(int)}(\mathbf{r}_1, t)$  is determined by equation (2.8) and

$$\hat{V}^{*(int)}(\mathbf{r}, t) = \exp(-i\hat{H}(\mathbf{r})t/\hbar) V(\mathbf{r}, t) \exp(i\hat{H}(\mathbf{r})t/\hbar). \tag{2.29}$$

The solution of equation (2.28) with initial conditions is equal to

$$\tilde{\psi}^*(\mathbf{r}_2, t) \tilde{\psi}(\mathbf{r}_1, t) = \hat{T} \exp\left(-\frac{i}{\hbar} \int_0^t d\tau (\hat{V}^{(int)}(\mathbf{r}_1, \tau) - \hat{V}^{*(int)}(\mathbf{r}_2, \tau))\right) \psi^{*(0)}(\mathbf{r}_2) \psi^{(0)}(\mathbf{r}_1) \tag{2.30}$$

which gives, analogously to (2.10)-(2.15), the averaged evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} \langle \tilde{\psi}^*(\mathbf{r}_2, t) \tilde{\psi}(\mathbf{r}_1, t) \rangle &= -\frac{1}{\hbar^2} \int_0^t d\tau (\langle \hat{V}^{(int)}(\mathbf{r}_1, t) \hat{V}^{(int)}(\mathbf{r}_1, \tau) \rangle + \langle \hat{V}^{*(int)}(\mathbf{r}_2, t) \hat{V}^{*(int)}(\mathbf{r}_2, \tau) \rangle \\ &\quad - \langle \hat{V}^{*(int)}(\mathbf{r}_2, t) \hat{V}^{(int)}(\mathbf{r}_1, \tau) \rangle - \langle \hat{V}^{(int)}(\mathbf{r}_1, t) \hat{V}^{*(int)}(\mathbf{r}_2, \tau) \rangle) \\ &\quad \times \langle \tilde{\psi}^*(\mathbf{r}_2, t) \tilde{\psi}(\mathbf{r}_1, t) \rangle \end{aligned} \tag{2.31}$$

or, after reciprocal transformation to  $\langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle$ ,

$$\begin{aligned} \frac{\partial}{\partial t} \langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle &= -\frac{i}{\hbar} (\hat{H}(\mathbf{r}_1) - \hat{H}(\mathbf{r}_2)) \langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle \\ &\quad - \frac{1}{\hbar^2} \int_0^t d\tau (\langle V(\mathbf{r}_1, t) \exp(-i\hat{H}(\mathbf{r}_1)(t-\tau)/\hbar) V(\mathbf{r}_1, \tau) \rangle \\ &\quad \times \exp(i\hat{H}(\mathbf{r}_1)(t-\tau)/\hbar) + \langle V(\mathbf{r}_2, t) \exp(i\hat{H}(\mathbf{r}_2)(t-\tau)/\hbar) V(\mathbf{r}_2, \tau) \rangle \\ &\quad \times \exp(-i\hat{H}(\mathbf{r}_2)(t-\tau)/\hbar) - \langle V(\mathbf{r}_1, t) \exp(i\hat{H}(\mathbf{r}_2)(t-\tau)/\hbar) V(\mathbf{r}_2, \tau) \rangle \\ &\quad \times \exp(-i\hat{H}(\mathbf{r}_2)(t-\tau)/\hbar) - \langle V(\mathbf{r}_2, t) \exp(-i\hat{H}(\mathbf{r}_1)(t-\tau)/\hbar) V(\mathbf{r}_1, \tau) \rangle \\ &\quad \times \exp(i\hat{H}(\mathbf{r}_1)(t-\tau)/\hbar)) \langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle. \end{aligned} \tag{2.32}$$

Although equation (2.32) was derived with the use of the factorised initial conditions (2.30), it is easy to see that the result does not change for initial conditions of more general type:

$$\langle \psi(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle|_{t=0} = \sum_{j,k} \rho_{jk}^{(0)} \psi_j^*(\mathbf{r}_2) \psi_k(\mathbf{r}_1) \tag{2.33}$$

where  $\rho_{jk}^{(0)}$  is a constant matrix.



### 2.3. Generalisation to quantised fluctuations of fields

Equation (2.32) is valid for classical fields  $V(\mathbf{r}, t)$ . The modification to quantised fields  $\hat{V}(\mathbf{r}, t)$  is very slight. In the spirit of secondary quantisation, the complex conjugate  $\psi^*(\mathbf{r}, t)$  should be replaced by the Hermitian conjugate  $\psi^\dagger(\mathbf{r}, t)$ . The corresponding Schrödinger equation has the form

$$-i\hbar \frac{\partial \psi^\dagger(\mathbf{r}, t)}{\partial t} = \psi^\dagger(\mathbf{r}, t) \hat{H}(\mathbf{r}) + \psi^\dagger(\mathbf{r}, t) \hat{V}(\mathbf{r}, t) \quad (2.34)$$

$$\psi^\dagger(\mathbf{r}, t) = \tilde{\psi}^\dagger(\mathbf{r}, t) \exp(i\hat{H}(\mathbf{r})t/\hbar) \quad (2.35)$$

$$-i\hbar \frac{\partial \tilde{\psi}^\dagger(\mathbf{r}, t)}{\partial t} = \tilde{\psi}^\dagger(\mathbf{r}, t) \hat{V}^{(\text{int})}(\mathbf{r}, t). \quad (2.36)$$

Here all operators act from right to left. The corresponding solution of equation (2.36) can be written with the use of the antichronologisation operator  $\hat{T}^+$  (cf Schwinger 1961, Keldysh 1964):

$$\tilde{\psi}^\dagger(\mathbf{r}, t) = \tilde{\psi}^{(0)\dagger}(\mathbf{r}) \hat{T}^+ \exp\left(i \int_0^t d\tau \hat{V}^{(\text{int})}(\mathbf{r}, \tau)\right). \quad (2.37)$$

The action of the operator  $\hat{T}^+$  is reciprocal to that of  $\hat{T}$  (see equation (2.11)). The averaging gives the evolution equation

$$\frac{\partial \langle \tilde{\psi}^\dagger(\mathbf{r}, t) \rangle}{\partial t} = -\frac{1}{\hbar^2} \langle \tilde{\psi}^\dagger(\mathbf{r}, t) \rangle \int_0^t d\tau \overline{\hat{V}^{(\text{int})}(\mathbf{r}, \tau) \hat{V}^{(\text{int})}(\mathbf{r}, t)}. \quad (2.38)$$

Thus, in the more exact form, the second and third averaged stochastic operators in equation (2.32) should be treated as operators of the type

$$\overline{\exp(-i\hat{H}(\mathbf{r}_2)(t-\tau)/\hbar) \hat{V}(\mathbf{r}_2, \tau) \exp(i\hat{H}(\mathbf{r}_2)(t-\tau)/\hbar) \hat{V}(\mathbf{r}_1, t)}$$

acting from right to left. It can be proved that this leads to permutation of creation and annihilation operators for the quantised field  $\hat{V}(\mathbf{r}, t)$  with respect to the usual order of action from left to right and ensures the asymptotic evolution to the thermodynamic density matrix in the case of coupling with thermoequilibrium fluctuations.

### 2.4. White-noise approximation

Equations (2.16) and (2.32) are simplified when the characteristic correlation time in the correlator (2.3) is much less than the eigenfrequencies of the operator  $\hat{H}(\mathbf{r})/\hbar$ . In this case all exponent operators may be neglected and the correlator (2.3) can be approximated by white-noise correlations:

$$\Gamma^{(0)}(\mathbf{r}_1 - \mathbf{r}_2) \delta(t_1 - t_2) \equiv \int_{-\infty}^{\infty} d\tau \Gamma(\mathbf{r}_1 - \mathbf{r}_2, \tau) \delta(t_1 - t_2). \quad (2.39)$$

The corresponding equations (2.16) and (2.32) are equal in this approximation to

$$\frac{\partial \langle \psi(\mathbf{r}, t) \rangle}{\partial t} = -\frac{i}{\hbar} \hat{H}(\mathbf{r}) \langle \psi(\mathbf{r}, t) \rangle - \frac{\Gamma^{(0)}(0)}{2\hbar^2} \langle \psi(\mathbf{r}, t) \rangle \quad (2.40)$$

$$\begin{aligned} & \frac{\partial}{\partial t} \langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle \\ &= -\frac{i}{\hbar} (\hat{H}(\mathbf{r}_1) - \hat{H}(\mathbf{r}_2)) \langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle \\ & \quad + \frac{1}{\hbar^2} (\Gamma^{(0)}(\mathbf{r}_1 - \mathbf{r}_2) - \Gamma^{(0)}(0)) \langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle. \end{aligned} \tag{2.41}$$

The last equation coincides with that obtained previously by Jayannavar and Kumar (1982).

The equations for higher-order equal time correlators and for correlators taken at different times can be derived analogously. We do not write them down explicitly because the algorithm is quite clear.

### 2.5. The response to an external non-stationary deterministic field

Here we consider the situation where a regular non-stationary part is added to the regular static potential in equation (2.1) so that the full regular deterministic potential and Hamiltonian are equal to

$$V^{(d)}(\mathbf{r}, t) = U(\mathbf{r}) + \tilde{V}^{(f)}(\mathbf{r}, t) \tag{2.42}$$

$$\hat{H}^{(d)}(\mathbf{r}, t) = \hat{H}(\mathbf{r}) + \tilde{V}^{(f)}(\mathbf{r}, t) \tag{2.43}$$

where  $\tilde{V}^{(f)}(\mathbf{r}, t)$  corresponds to a regular non-stationary external field. In this case the Hamiltonian  $\hat{H}(\mathbf{r})$  in equations (2.16) and (2.32) must be replaced by  $\hat{H}^{(d)}(\mathbf{r}, t)$  and the operators  $\exp(-i\hat{H}(\mathbf{r})(t-\tau)/\hbar)$ ,  $\exp(i\hat{H}(\mathbf{r})(t-\tau)/\hbar)$  must be replaced by evolution operators,

$$\hat{U}^{(d)}(\mathbf{r}; t, \tau) = \hat{T} \exp\left(-i \int_{\tau}^t d\tau' \hat{H}^{(d)}(\mathbf{r}, \tau')/\hbar\right) \tag{2.44}$$

and  $\hat{U}^{(d)\dagger}(\mathbf{r}; t, \tau)$ , where  $\hat{U}^{(d)\dagger}(\mathbf{r}; t, \tau)$  is the Hermitian conjugate to  $\hat{U}^{(d)}(\mathbf{r}; t, \tau)$  and  $\hat{T}$  corresponds to chronologisation ordering. If the field is weak and  $\langle \tilde{V}^{(f)}(\mathbf{r}, t) \rangle_{\tau_{\text{corr}}} \ll \hbar$ , then the deterministic evolution operators can be approximated by

$$\begin{aligned} \hat{U}^{(d)}(\mathbf{r}; t, \tau) \approx & \exp(-i\hat{H}(\mathbf{r})t/\hbar) \left( 1 - \frac{i}{\hbar} \int_{\tau}^t d\tau' \exp(i\hat{H}(\mathbf{r})\tau'/\hbar) \right. \\ & \left. \times V^{(f)}(\mathbf{r}, \tau') \exp(-i\hat{H}(\mathbf{r})\tau'/\hbar) \right) \exp(i\hat{H}(\mathbf{r})\tau/\hbar) \end{aligned} \tag{2.45}$$

$$\begin{aligned} \hat{U}^{(d)\dagger}(\mathbf{r}; t, \tau) \approx & \exp(-i\hat{H}(\mathbf{r})\tau/\hbar) \left( 1 + \frac{i}{\hbar} \int_{\tau}^t d\tau' \exp(i\hat{H}(\mathbf{r})\tau'/\hbar) \right. \\ & \left. \times V^{(f)}(\mathbf{r}, \tau') \exp(-i\hat{H}(\mathbf{r})\tau'/\hbar) \right) \exp(i\hat{H}(\mathbf{r})t/\hbar). \end{aligned} \tag{2.46}$$

The formulae (2.45) and (2.46) are solutions to the problem of linear response to a weak non-stationary regular external field.

### 3. Example: coupling with phonons

Stochastic evolution should be subject to some general physical principles. For example, the classical Fokker-Planck equation describing the motion of a particle

coupled with an equilibrium thermostat must give, in asymptotics, the Maxwell distribution. This condition restricts the possible form of the equations. If, e.g., diffusion in momentum space is defined, then viscous terms are determined unambiguously. The same question must be asked with respect to equation (2.32). Let the correlators  $\langle \hat{V}(\mathbf{r}_1, t) \hat{V}(\mathbf{r}_2, \tau) \rangle$  correspond to thermoequilibrium fluctuations of the fields. Does the asymptotic stationary solution of equation (2.32) coincide with the thermodynamic density matrix?

Despite the generality of the question, it is better to consider first a particular example, while the general proof will be given in the next section. We shall consider in this section the well studied problem of interaction with longitudinal phonons. The fluctuating potential in equation (2.1) is equal in this case to (see, e.g., Abrikosov *et al* 1963, Kittel 1963)

$$\hat{V}(\mathbf{r}, t) = w \operatorname{div} \hat{\mathbf{u}}(\mathbf{r}, t). \quad (3.1)$$

The constant  $w$  is of the order of the Fermi energy for metals and the Debye energy for dielectrics. Displacements of the medium,  $\hat{\mathbf{u}}(\mathbf{r}, t)$ , are quantised and given by

$$\hat{\mathbf{u}}(\mathbf{r}, t) = \Omega^{-1/2} \sum_{\mathbf{q}} \frac{\mathbf{q}}{|\mathbf{q}|} \left( \frac{\hbar}{2\rho\omega_{\mathbf{q}}} \right)^{1/2} (c_{\mathbf{q}}^+ \exp(i\omega_{\mathbf{q}}t - i\mathbf{q}\mathbf{r}) + c_{\mathbf{q}} \exp(-i\omega_{\mathbf{q}}t + i\mathbf{q}\mathbf{r})) \quad (3.2)$$

where  $\Omega$  is the volume of a pattern,  $\rho$  is the mass density of a medium,  $\omega_{\mathbf{q}} = s|\mathbf{q}|$  ( $s$  is the velocity of sound), and  $c_{\mathbf{q}}^+$  and  $c_{\mathbf{q}}$  are creation and annihilation operators with commutator

$$[c_{\mathbf{q}}, c_{\mathbf{q}'}^+] = \delta_{\mathbf{q}\mathbf{q}'}. \quad (3.3)$$

A phonon system is assumed to be in thermodynamic equilibrium and has the Planck distribution function:

$$\langle c_{\mathbf{q}}^+ c_{\mathbf{q}} \rangle \equiv N_{\mathbf{q}} = [\exp(\beta\hbar\omega_{\mathbf{q}}) - 1]^{-1} \quad (3.4)$$

and a particle is assumed to be free (i.e.  $U(\mathbf{r}) = 0$  in equation (2.1)).

The corresponding correlators in equations (2.16) and (2.32) are calculated with the use of the identity

$$\begin{aligned} & \exp(-i\hat{H}(\mathbf{r})t/\hbar) \exp(i\mathbf{q}\mathbf{r}) \exp(i\hat{H}(\mathbf{r})t/\hbar) \\ &= \exp(i\mathbf{q}\mathbf{r}) \exp(-i(\hbar^2\mathbf{q}^2/2m + \hbar\mathbf{q} \cdot \hat{\mathbf{p}}/m)t/\hbar) \end{aligned} \quad (3.5)$$

where  $\hat{\mathbf{p}} = -i\hbar\partial/\partial\mathbf{r}$  is the momentum operator (note that the equality (3.5) is valid only for  $U(\mathbf{r}) = 0$ ). Then equation (2.16) for the averaged  $\psi$  function is given by

$$\begin{aligned} \frac{\partial\langle\psi(\mathbf{r}, t)\rangle}{\partial t} &= \frac{i\hbar}{2m} \frac{\partial^2\langle\psi(\mathbf{r}, t)\rangle}{\partial\mathbf{r}^2} \\ &- \frac{1}{\hbar^2} \int_0^t d\tau \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{w^2\hbar\mathbf{q}^2}{2\rho\omega_{\mathbf{q}}} [N_{\mathbf{q}} \exp(i\omega_{\mathbf{q}}(t-\tau)) \\ &\times \exp(-i(\hbar^2\mathbf{q}^2/2m + \hbar\mathbf{q} \cdot \hat{\mathbf{p}}/m)(t-\tau)/\hbar) + (N_{\mathbf{q}} + 1) \exp(-i\omega_{\mathbf{q}}(t-\tau)) \\ &\times \exp(-i(\hbar^2\mathbf{q}^2/2m - \hbar\mathbf{q} \cdot \hat{\mathbf{p}}/m)(t-\tau)/\hbar)] \langle\psi(\mathbf{r}, t)\rangle. \end{aligned} \quad (3.6)$$

For large times  $t \gg (\omega_{\mathbf{q}})^{-1} \sim \hbar/k_{\text{B}}T$  the upper limit of integration can be replaced approximately by infinity and the asymptotic solution can be of the form

$$\langle\psi(\mathbf{r}, t)\rangle = \sum_{\mathbf{p}} \langle a_{\mathbf{p}} \rangle \exp(-i\varepsilon_{\mathbf{p}}t/\hbar - \Gamma_{\mathbf{p}}t + i\mathbf{p}\mathbf{r}) \quad (3.7)$$

where the function  $\langle a_p \rangle$  is determined by the initial conditions. Using the formulae

$$\int_0^\infty dt \exp(\mp i \varepsilon t / \hbar) = \pi \hbar \delta(\varepsilon) \mp i \hbar \hat{P} / \varepsilon \tag{3.8}$$

where the operator  $\hat{P}$  means that integration in the vicinity of a singularity must be performed in the principal-value sense, one obtains

$$\tilde{\varepsilon}_p = \varepsilon_p + \Delta \varepsilon_p \tag{3.9}$$

$$\varepsilon_p = p^2 / 2m \tag{3.10}$$

$$\Delta \varepsilon_p = \int \frac{d^3 q}{(2\pi)^3} \int d^3 p' \frac{w^2 \hbar q^2}{2\rho\omega_q} \left( \frac{N_q \delta(\hbar q + p - p')}{\varepsilon_p + \hbar\omega_q - \varepsilon_{p'}} + \frac{(N_q + 1) \delta(\hbar q + p' - p)}{\varepsilon_p - \hbar\omega_q - \varepsilon_{p'}} \right) \tag{3.11}$$

$$\begin{aligned} \Gamma_p &= \frac{\pi}{\hbar} \int \int \frac{d^3 q}{(2\pi)^3} d^3 p' \frac{w^2 \hbar q^2}{2\rho\omega_q} \\ &\quad \times [\delta(p - p' - \hbar q) \delta(\varepsilon_p - \varepsilon_{p'} - \hbar\omega_q) (N_q + 1) \\ &\quad + \delta(p + \hbar q - p') \delta(\varepsilon_p + \hbar\omega_q - \varepsilon_{p'}) N_q] \\ &= \frac{w^2 m}{8\pi \hbar \rho s |p|} \left( \theta(|p|/m - s) \int_0^{2m(|p|/m - s)/\hbar} dq q^2 (N_q + 1) \right. \\ &\quad \left. + \int_{\max\{2m(s - |p|/m)/\hbar, 0\}}^\infty dq q^2 N_q \right) \tag{3.12} \end{aligned}$$

where  $\theta(x)$  is the Heaviside step function. As has been pointed out above, the integration in equation (3.11) must be understood in the principal-value sense (in more realistic calculations one must also include finite damping of phonons). At zero temperature ( $N_q = 0$ ) the expression for  $\Gamma_p$  coincides with that obtained previously by Migdal (1958) with the use of Green functions.

The corresponding equation (2.32) for the density matrix in this model is equal to

$$\begin{aligned} &\frac{\partial}{\partial t} \langle \psi^*(r_2, t) \psi(r_1, t) \rangle \\ &= -\frac{i}{2m\hbar} (\hat{p}_1^2 - \hat{p}_2^2) \langle \psi^*(r_2, t) \psi(r_1, t) \rangle \\ &\quad - \frac{1}{\hbar^2} \int_0^t d\tau \int \frac{d^3 q}{(2\pi)^3} \frac{w^2 \hbar q^2}{2\rho\omega_q} \{ N_q [1 - \exp(iq(r_1 - r_2))] \\ &\quad \times [\exp(i\omega_q(t - \tau)) \exp(-i(\varepsilon_q + \hbar q \cdot \hat{p}_1/m)(t - \tau)/\hbar) \\ &\quad + \exp(-i\omega_q(t - \tau)) \exp(i(\varepsilon_q - \hbar q \cdot \hat{p}_2/m)(t - \tau)/\hbar)] \\ &\quad + (N_q + 1) [1 - \exp(iq(r_2 - r_1))] \\ &\quad \times [\exp(-i\omega_q(t - \tau)) \exp(-i(\varepsilon_q - \hbar q \cdot \hat{p}_1/m)(t - \tau)/\hbar) \\ &\quad + \exp(i\omega_q(t - \tau)) \exp(i(\varepsilon_q + \hbar q \cdot \hat{p}_2/m)(t - \tau)/\hbar)] \} \\ &\quad \times \langle \psi^*(r_2, t) \psi(r_1, t) \rangle \tag{3.13} \end{aligned}$$

where  $\hat{p}_1 = -i\hbar\partial/\partial r_1$ ,  $\hat{p}_2 = -i\hbar\partial/\partial r_2$ ,  $\varepsilon_q = \hbar^2 q^2 / 2m$ . We describe here only the asymptotic (for  $t \gg \hbar/k_B T$ ) evolution of plane wavepackets:

$$\langle \psi^*(r_2, t) \psi(r_1, t) \rangle = \sum_p n_p(t) \exp(ip(r_1 - r_2)/\hbar) \tag{3.14}$$

where  $n_p(t)$  changes slowly on the timescale  $\sim \hbar/k_B T$ . Using equation (3.8) it is easy to obtain

$$\begin{aligned} \frac{\partial n_p(t)}{\partial t} = \frac{2\pi}{\hbar} \int \int \frac{d^3 q}{(2\pi)^3} d^3 p' \frac{w^2 \hbar q^2}{2\rho\omega_q} \\ \times \{ \delta(\mathbf{p} - \mathbf{p}' - \hbar\mathbf{q}) \delta(\varepsilon_p - \varepsilon_{p'} - \hbar\omega_q) [n_p(t)N_q - n_p(t)(N_q + 1)] \\ + \delta(\mathbf{p} + \hbar\mathbf{q} - \mathbf{p}') \delta(\varepsilon_p + \hbar\omega_q - \varepsilon_{p'}) [n_p(t)(N_q + 1) - n_p(t)N_q] \} \end{aligned} \quad (3.15)$$

which coincides with the quantum kinetic equation. Terms with  $(N_q + 1)$  correspond to emission of a phonon, while terms with  $N_q$  describe absorption of a phonon. It is worth noting that renormalisation corrections to the energy spectrum of a particle (see equations (3.9)–(3.11)) mutually cancel exactly in the kinetic equation (3.15). For an energetic particle with  $|\mathbf{p}|/m \gg s$  the transfers of momentum are small and equation (3.15) may be approximated by the Fokker–Planck equation in momentum space. The mean balance of energy transfer is given by

$$\begin{aligned} \frac{\partial \langle E \rangle}{\partial t} = \frac{\partial}{\partial t} \int d^3 p n_p(t) \varepsilon_p \\ = \frac{2\pi}{\hbar} \int \frac{d^3 q}{(2\pi)^3} d^3 p d^3 p' \frac{w^2 \hbar q^2}{2\rho\omega_q} \delta(\mathbf{p} + \hbar\mathbf{q} - \mathbf{p}') \\ \times \delta(\varepsilon_p + \hbar\omega_q - \varepsilon_{p'}) \hbar\omega_q [n_p(t)N_q - n_p(t)(N_q + 1)]. \end{aligned} \quad (3.16)$$

As is seen from equation (3.16), the energy is always lost at zero temperature ( $N_q = 0$ ). This can be interpreted as ‘freezing’ of the plane wavepacket. The energy loss at zero temperature is due only to supersonic harmonics of the wavepacket with  $|\mathbf{p}|/m > s$ . However, this result does not exclude the possibility of non-trivial dynamics with respect to the centre of mass coordinate  $(\mathbf{r}_1 + \mathbf{r}_2)/2$  even at zero temperature (cf Caldeira and Leggett 1983a, Riseborough *et al* 1985, Haake and Reibold 1985). However, this problem needs further detailed investigation. The equilibrium solution of equation (3.15) is equal to

$$n_p = Z^{-1} \exp(-\beta p^2/2m) \quad (3.17)$$

and exactly coincides with the thermodynamic density matrix. Here  $Z$  is the partition function.

Comparing equations (3.12) and (3.15) we conclude that the dephasing time for the  $\psi$  function is of the same order as the relaxation time of the distribution function.

This particular example illustrates clearly the nature of coupling with an external medium and the character of the asymptotic evolution to thermoequilibrium. We shall show in the next section that such a situation holds generally.

## 4. General proof of asymptotic evolution to thermoequilibrium

### 4.1. Evolution of the density matrix in eigenstate representation

The general proof of the asymptotic evolution of the equal time correlator  $\langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle$  to the thermodynamic density matrix for a system coupled to given thermoequilibrium fluctuations of fields can be obtained by projection of equation (2.32) on the space of eigenfunctions of the Hamiltonian  $\hat{H}(\mathbf{r})$  (see equation (2.1)):

$$\langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle = \sum_{m,n} a_{mn}(t) \tilde{\psi}_n^*(\mathbf{r}_2) \psi_m(\mathbf{r}_1). \quad (4.1)$$

It is assumed that the eigenfunctions  $\{\psi_m(\mathbf{r})\}$  correspond to a complete orthonormal set. The spectrum is chosen for simplicity to be discrete (this means, e.g., that a free particle is confined in a large box, etc). Using the well known bra and ket representation for matrix elements of the products of operators,

$$\langle n|\hat{A}\hat{B}|m\rangle = \sum_j \langle n|\hat{A}|j\rangle\langle j|\hat{B}|m\rangle \tag{4.2}$$

the final answer can be written in the form

$$\begin{aligned} \frac{da_{mn}(t)}{dt} = & -\frac{i}{\hbar}(E_m - E_n)a_{mn}(t) \\ & -\frac{1}{\hbar^2} \sum_{j,k} \int_0^t d\tau \Gamma_{jj,km}^{(+)}(t-\tau) \exp(-i(E_j - E_k)(t-\tau)/\hbar) a_{kn}(t) \\ & -\frac{1}{\hbar^2} \sum_{j,k} \int_0^t d\tau \Gamma_{kn,jj}^{(-)}(\tau-t) \exp(-i(E_k - E_j)(t-\tau)/\hbar) a_{mk}(t) \\ & +\frac{1}{\hbar^2} \sum_{k,p} \int_0^t d\tau \Gamma_{mn,kp}^{(+)}(t-\tau) \exp(-i(E_m - E_k)(t-\tau)/\hbar) a_{kp}(t) \\ & +\frac{1}{\hbar^2} \sum_{k,p} \int_0^t d\tau \Gamma_{pk,nm}^{(-)}(\tau-t) \exp(-i(E_p - E_n)(t-\tau)/\hbar) a_{kp}(t) \end{aligned} \tag{4.3}$$

where

$$\Gamma_{mn,pq}^{(+)}(t-\tau) = \iint d^d\mathbf{r}_1 d^d\mathbf{r}_2 \psi_m^*(\mathbf{r}_1)\psi_n(\mathbf{r}_2)\langle \hat{V}(\mathbf{r}_2, t)\hat{V}(\mathbf{r}_1, \tau)\rangle\psi_p(\mathbf{r}_1)\psi_q^*(\mathbf{r}_2) \tag{4.4}$$

$$\Gamma_{mn,pq}^{(-)}(\tau-t) = \iint d^d\mathbf{r}_1 d^d\mathbf{r}_2 \psi_m^*(\mathbf{r}_1)\psi_n(\mathbf{r}_2)\langle \hat{V}(\mathbf{r}_1, \tau)\hat{V}(\mathbf{r}_2, t)\rangle\psi_p(\mathbf{r}_1)\psi_q^*(\mathbf{r}_2). \tag{4.5}$$

We shall now consider the special case of coupling with a thermoequilibrium thermostat.

According to the fluctuation-dissipation theorem (Kubo 1957, 1966, Martin and Schwinger 1959) the correlators  $\langle \hat{V}(\mathbf{r}_1, \tau)\hat{V}(\mathbf{r}_2, t)\rangle$  and  $\langle \hat{V}(\mathbf{r}_2, t)\hat{V}(\mathbf{r}_1, \tau)\rangle$  are related in thermoequilibrium by

$$\begin{aligned} \langle \hat{V}(\mathbf{r}_2, t)\hat{V}(\mathbf{r}_1, \tau)\rangle &= \langle \hat{V}(\mathbf{r}_1, \tau)\hat{V}(\mathbf{r}_2, t+i\beta\hbar)\rangle \\ &= \exp(i\beta\hbar\partial/\partial t)\langle \hat{V}(\mathbf{r}_1, \tau)\hat{V}(\mathbf{r}_2, t)\rangle. \end{aligned} \tag{4.6}$$

Here  $\beta$  is, as usual, the reciprocal temperature times the Boltzmann constant.

We should add to this general result two auxiliary equalities:

$$\langle \hat{V}(\mathbf{r}_2, t)\hat{V}(\mathbf{r}_1, t)\rangle = \langle \hat{V}(\mathbf{r}_1, t)\hat{V}(\mathbf{r}_2, t)\rangle \tag{4.7}$$

$$\left(\frac{\partial}{\partial t}\right)^{2k-1} (\langle \hat{V}(\mathbf{r}_2, t)\hat{V}(\mathbf{r}_1, \tau)\rangle + \langle \hat{V}(\mathbf{r}_1, \tau)\hat{V}(\mathbf{r}_2, t)\rangle)|_{t=\tau} = 0 \quad k = 1, 2, \dots \tag{4.8}$$

Equality (4.7) corresponds to the causality principle, while equality (4.8) corresponds to the evenness of the quantum correlation function with respect to the difference of times  $(t - \tau)$  under stationary conditions.

Using equations (4.6)–(4.8) and integrating by parts in the fourth and fifth terms in equation (4.3), it can be directly checked that the asymptotic stationary solution is equal to

$$a_{mn}^{(e)} = \delta_{mn} e^{-\beta E_m} / Z \tag{4.9}$$

where  $\delta_{mn}$  is the Kronecker symbol and  $Z$  is the partition function

$$Z = \sum_m e^{-\beta E_m}. \tag{4.10}$$

This means that the asymptotic solution of equation (2.32) coincides with the thermodynamic density matrix (see, e.g., Landau and Lifshitz 1969, Feynman 1972):

$$\langle \psi^*(\mathbf{r}_2) \psi(\mathbf{r}_1) \rangle_T = Z^{-1} \sum_j e^{-\beta E_j} \psi_j^*(\mathbf{r}_2) \psi_j(\mathbf{r}_1). \tag{4.11}$$

We should stress that the stationary solution (4.9) is valid only for coupling with thermoequilibrium fluctuations. However, equation (4.3) can be applied also to arbitrary non-equilibrium Gaussian noise.

It follows from this result that any non-trivial stationary dynamics at zero temperature is permissible if and only if the ground state is degenerate. For example, this means that in order to obtain non-trivial Brownian motion the sizes of a system must satisfy the condition  $L \gg \lambda_T \sim \hbar / (mk_B T)^{1/2}$  (where  $L$  is the characteristic length of a box and  $\lambda_T$  is the thermal wavelength of a particle) and, thus, the sizes of a system must tend to infinity at zero temperature. In this limit the ground state will become degenerate in view of the exact translational invariance of the infinite system.

#### 4.2. Dispersion relationships for frequency shifts and broadenings

In the limit  $t \gg \tau_{\text{corr}}$ , where  $\tau_{\text{corr}}$  is the characteristic correlation time of a noise (for thermoequilibrium fluctuations  $\tau_{\text{corr}} \sim \hbar\beta$ ), the system (4.3) can be simplified. Using Fourier transformation of the matrix elements (4.4) and (4.5)

$$\Gamma_{mn,pq}^{(\pm)}(\pm(t - \tau)) = \int \frac{d\omega}{2\pi} \Gamma_{mn,pq}^{(\pm)}(\omega) \exp(-i\omega(t - \tau)) \tag{4.12}$$

and (3.8) one obtains immediately:

$$\begin{aligned} \frac{da_{mn}(t)}{dt} = & -\frac{i}{\hbar} (E_m - E_n) a_{mn}(t) + \frac{1}{\hbar} \sum_{j,k} (i\Delta E_{jj,km}^{(+)} - B_{jj,km}^{(+)}) a_{kn}(t) \\ & + \frac{1}{\hbar} \sum_{j,k} (i\Delta E_{kn,ji}^{(-)} - B_{kn,ji}^{(-)}) a_{mk}(t) \\ & - \frac{1}{\hbar} \sum_{k,p} (i\Delta E_{mn,kp}^{(+)} - B_{mn,kp}^{(+)}) a_{kp}(t) \\ & - \frac{1}{\hbar} \sum_{k,p} (i\Delta E_{pk,nm}^{(-)} - B_{pk,nm}^{(-)}) a_{kp}(t) \end{aligned} \tag{4.13}$$

where

$$\Delta E_{mn,pq}^{(\pm)} = \int \frac{d\omega}{2\pi} \frac{\Gamma_{mn,pq}^{(\pm)}(\omega)}{\hbar\omega + E_m - E_p} \tag{4.14}$$

$$B_{mn,pq}^{(\pm)} = \frac{1}{2\hbar} \Gamma_{mn,pq}^{(\pm)}((E_p - E_m)/\hbar). \tag{4.15}$$

If the broadenings  $B_{mn,pq}$  are considered as functions of  $(E_p - E_m)/\hbar$  then frequency

shifts and broadenings are related by (cf Riseborough *et al* 1985)

$$\Delta E_{mn,pq}^{(\pm)} = \frac{1}{\pi} \int d\omega \frac{B_{mn,pq}^{(\pm)}(\omega)}{\omega + (E_m - E_p)/\hbar}. \tag{4.16}$$

It follows directly from definitions (4.4), (4.5) and (4.12) that

$$\Gamma_{mn,pq}^{(+)}(-\omega) = \Gamma_{qp,nm}^{(-)}(\omega). \tag{4.17}$$

On the other hand, the fluctuation-dissipation theorem (4.6) gives the relationship

$$\Gamma_{mn,pq}^{(+)}(\omega) = e^{\beta\hbar\omega} \Gamma_{mn,pq}^{(-)}(\omega). \tag{4.18}$$

Using equations (4.17) and (4.18) the asymptotic stationary solution (4.9) can be again easily reproduced.

Although static thermoequilibrium properties are determined by unrenormalised energy levels, in view of the exact cancellation of shifts  $\Delta E_{mn,pq}$  for diagonal elements of the density matrix (see equation (4.9)), the dynamic properties will be determined by the off-diagonal elements  $a_{mn}(t)$  with renormalised energies. These shifts in eigenfrequencies due to coupling with the thermostat can be measured, e.g., with the use of resonance response to a non-stationary external field (see § 2.5). We should note also that in the Markovian approximation (2.39) frequency shifts are absent.

### 4.3. Exact conservation of normalisation during stochastic evolution

The analogous asymptotic stationary solution for the equal time correlator  $\langle \psi^*(\mathbf{r}_4, t) \psi^*(\mathbf{r}_3, t) \psi(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle$  is given by

$$\begin{aligned} & \langle \psi^*(\mathbf{r}_4) \psi^*(\mathbf{r}_3) \psi(\mathbf{r}_2) \psi(\mathbf{r}_1) \rangle_T \\ &= \langle \psi^*(\mathbf{r}_4) \psi(\mathbf{r}_2) \rangle_T \langle \psi^*(\mathbf{r}_3) \psi(\mathbf{r}_1) \rangle_T + \langle \psi^*(\mathbf{r}_4) \psi(\mathbf{r}_1) \rangle_T \langle \psi^*(\mathbf{r}_3) \psi(\mathbf{r}_2) \rangle_T \\ & \quad - Z^{-2} \sum_j e^{-2\beta E_j} \psi_j^*(\mathbf{r}_4) \psi_j^*(\mathbf{r}_3) \psi_j(\mathbf{r}_2) \psi_j(\mathbf{r}_1). \end{aligned} \tag{4.19}$$

Here  $\langle \psi^*(\mathbf{r}_4) \psi(\mathbf{r}_1) \rangle_T$  is defined in equation (4.11).

It can be easily proved that the integrals

$$\begin{aligned} & \int d^d \mathbf{r} \langle \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t) \rangle \\ & \iint d^d \mathbf{r}_1 d^d \mathbf{r}_2 \langle \psi^*(\mathbf{r}_1, t) \psi(\mathbf{r}_1, t) \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_2, t) \rangle \end{aligned}$$

etc, do not depend on time and are conserved.

Hence, one may always define

$$\int d^d \mathbf{r} \langle \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t) \rangle = 1. \tag{4.20}$$

For such a normalisation, all equal time correlators are related by reduction equalities of the type

$$\begin{aligned} & \langle \psi^*(\mathbf{r}_n, t) \dots \psi^*(\mathbf{r}_k, t) \psi(\mathbf{r}_{k-1}, t) \dots \psi(\mathbf{r}_1, t) \rangle \\ &= \int d^d \mathbf{r} \langle \psi^*(\mathbf{r}_n, t) \dots \psi^*(\mathbf{r}_k, t) \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t) \psi(\mathbf{r}_{k-1}, t) \dots \psi(\mathbf{r}_1, t) \rangle \end{aligned} \tag{4.21}$$

which corresponds to the exact conservation of normalisation during stochastic evolution. In fact, this is a direct consequence of equation (2.1) with arbitrary potential  $V(\mathbf{r}, t)$ . The same identities will hold also for stationary thermoequilibrium correlators (cf equations (4.11) and (4.19)).



## 5. Two-level systems

The system (4.3) can be applied to various problems related to transitions in a discrete spectrum under action of both thermoequilibrium and non-equilibrium fluctuations of fields. We shall consider in this section the most elementary two-level system, again coupled with phonons. The typical physical realisation of this situation is given by glasses at low temperatures (Phillips 1972, Anderson *et al* 1972).

Using equations (3.2), (4.3)-(4.5) one obtains after elementary calculation

$$\frac{da_{11}}{dt} = (\Gamma_{22,11}^{(q)} + \Gamma_{11,22}^{(-q)})[a_{22}(N(\varepsilon) + 1) - a_{11}N(\varepsilon)] \quad (5.1)$$

$$\begin{aligned} \frac{da_{12}}{dt} = \frac{i}{\hbar} \varepsilon a_{12} - \Gamma_{22,11}^{(q)}(2N(\varepsilon) + 1)a_{12} \\ + \Gamma_{12,21}^{(-q)}(2N(\varepsilon) + 1)a_{21} + (\Gamma_{12,22}^{(-q)} - \Gamma_{11,21}^{(-q)})[a_{22}(N(\varepsilon) + 1) - a_{11}N(\varepsilon)] \end{aligned} \quad (5.2)$$

$$a_{11} + a_{22} = 1 \quad (5.3)$$

$$a_{12} = a_{21}^* \quad (5.4)$$

where

$$\begin{aligned} \Gamma_{mn,pk}^{(q)} = \frac{\pi}{\hbar} \int \int d^d \mathbf{r}_1 d^d \mathbf{r}_2 \psi_m^*(\mathbf{r}_1) \psi_n(\mathbf{r}_2) \\ \times \left( \frac{w^2 \hbar q^2}{2\rho\omega_q} \right) \left( \frac{4\pi q^2}{(2\pi)^3 s \hbar} \right) \exp(i\mathbf{q}(\mathbf{r}_1 - \mathbf{r}_2)) \psi_p(\mathbf{r}_1) \psi_k^*(\mathbf{r}_2) \end{aligned} \quad (5.5)$$

$$N(\varepsilon) = (e^{\beta\varepsilon} - 1)^{-1} \quad (5.6)$$

$$\varepsilon = E_2 - E_1 \quad \hbar\omega_q = \hbar s|\mathbf{q}| = \varepsilon. \quad (5.7)$$

We neglect for simplicity the renormalisation of energies and include only dephasing effects. Though the general solution of the system (5.1)-(5.4) can be found easily, the equations may be simplified even more as the following matrix elements are approximately equal to each other with a good accuracy:

$$\Gamma_{22,11}^{(q)} \approx \Gamma_{11,22}^{(-q)} \approx \Gamma_{12,21}^{(-q)} \equiv \Gamma(\varepsilon) \quad (5.8)$$

$$\Gamma_{12,22}^{(q)} \approx \Gamma_{11,21}^{(q)}. \quad (5.9)$$

Then diagonal and off-diagonal terms  $a_{mn}$  become decoupled. The solution can be of the form

$$a_{mn} = a_{mn}^{(e)} + b_{mn} e^{\gamma t} \quad (5.10)$$

where the equilibrium density matrix  $a_{mn}^{(e)}$  is given by

$$a_{mn}^{(e)} = \delta_{mn} e^{-\beta E_n} / Z \quad Z = 1 + e^{\beta\varepsilon} \quad (5.11)$$

(the energy is counted from the lower level  $E_1$ ) and  $b_{mn}$  depends on the initial conditions.

The decrement  $\gamma^{(d)}$  for the diagonal elements is equal to

$$\gamma^{(d)}(\varepsilon) = -2\Gamma(\varepsilon)(2N(\varepsilon) + 1) \equiv -2\gamma^{(0)}(\varepsilon) \quad (5.12)$$

while the corresponding decrements for the off-diagonal elements are given by

$$\gamma^{(\text{off})}(\varepsilon) = -\gamma^{(0)}(\varepsilon) \pm (\gamma^{(0)2}(\varepsilon) - \varepsilon^2 / \hbar^2)^{1/2}. \quad (5.13)$$

Thus the diagonal elements always relax to thermoequilibrium monotonically. The off-diagonal terms may oscillate if  $\gamma^{(0)}(\varepsilon) < \varepsilon / \hbar$ . This corresponds to so-called Rabi oscillations and is typical of quantum two-level systems (see, e.g., Landau and Lifshitz

1976). In the limit  $\gamma^{(0)}(\varepsilon) \gg \varepsilon/\hbar$  one of the off-diagonal decrements becomes small ( $\gamma^{(\text{off})} \approx \varepsilon^2/2\hbar^2\gamma^{(0)}(\varepsilon)$ ). This reproduces the well known effect of suppression of tunnelling by intensive noise, which was first pointed out by Simonius (1978) (see also Pomeau and Pumir 1985).

The observable macroscopic behaviour of the bulk pattern will be obtained by averaging the evolution equations on the distribution function of the level spacings  $g(\varepsilon)$ , i.e.

$$\langle e^{\gamma^{(\varepsilon)}t} \rangle = \int_0^\infty d\varepsilon g(\varepsilon) e^{\gamma^{(\varepsilon)}t}. \tag{5.14}$$

If the wavelengths of the phonons are much more than the length of localisation of the two-level system,  $|q|l_{\text{loc}} \ll 1$ , then  $\exp(iqr)$  in equation (5.5) can be expanded on the degrees of  $(qr)$ . This corresponds to the multipole expansion. The matrix elements

$$\int d^d r \psi_m^*(r) r \psi_m(r) = 0 \tag{5.15}$$

will be equal to zero due to the parity selection rule. Then to the lowest order (see equations (5.5), (5.6), (5.8) and (5.12)) one obtains  $\Gamma(\varepsilon) \propto \varepsilon^5$  and  $\gamma^{(d)} \propto \varepsilon^4$  at small energies. On the other hand, the density of states  $g(\varepsilon)$  is usually not equal to zero at  $\varepsilon = 0$  (Phillips 1972, Anderson *et al* 1972). In this case the averaging (5.14) gives the power law relaxation for large times for diagonal elements:

$$\langle \exp(\gamma^{(d)}(\varepsilon)t) \rangle \propto t^{-1/4}. \tag{5.16}$$

As has been shown in § 3, the same behaviour will be typical for the dephasing times of the localised wavefunctions, but the influence of the Simonius effect may partially change the situation.

In the short wavelength limit,  $q(r_1 - r_2) \gg 1$ , the matrix element  $\Gamma(\varepsilon) \propto \varepsilon^3$  and  $\gamma^{(d)}(\varepsilon) \propto \varepsilon^2 k_B T$ , which gives the relaxation law proportional to  $t^{-1/2}$ .

We have described here only the relaxational process due to phonon emission. In experimental situations, other relaxational mechanisms dominate usually. They are related to the structural reconstructions due to the overcoming of small energy barriers. The corresponding relaxation times are given by

$$\tau(\varepsilon) = \tau_0 e^{\beta\varepsilon} \tag{5.17}$$

where  $\tau_0 \sim \hbar/k_B\theta_D$  ( $\theta_D$  is the Debye temperature). After averaging on the level spacings, the asymptotic result is equal to

$$\langle e^{-t/\tau(\varepsilon)} \rangle \propto (\ln t)^{-1}. \tag{5.18}$$

Comparing (5.16) and (5.18) we see that the asymptotic tails will be logarithmic at the largest times in accordance with experiment. A more detailed study of relaxation processes and comparison with experiment is, however, outside the aim of this paper.

## 6. The quasiclassical Fokker-Planck equations

At high temperatures the quantum effects become less important and motion of a particle will be quasiclassical. We shall show that in this limit the quantum equation (2.32) for the density matrix may be approximated by the quasiclassical Fokker-Planck equation.

The first simplifications are related to the shortening of correlation times at high temperatures. If  $\tau_{\text{corr}}^{-1} \gg \langle \hat{H}(r) \rangle / \hbar$  then the corresponding correlator operators in

equation (2.32) can be expanded in terms of the parameter  $\langle \hat{H}(\mathbf{r}) \rangle_{\tau_{\text{corr}}} / \hbar \ll 1$ , i.e.

$$\begin{aligned} & \langle \hat{V}(\mathbf{r}_2, t) \exp(-i\hat{H}(\mathbf{r}_1)(t-\tau)/\hbar) \hat{V}(\mathbf{r}_1, \tau) \exp(i\hat{H}(\mathbf{r}_1)(t-\tau)/\hbar) \rangle \\ & \approx \langle \hat{V}(\mathbf{r}_2, t) \hat{V}(\mathbf{r}_1, \tau) \rangle - \frac{i}{\hbar} \langle \hat{V}(\mathbf{r}_2, t) [\hat{H}(\mathbf{r}_1), \hat{V}(\mathbf{r}_1, \tau)] (t-\tau) \rangle. \end{aligned} \quad (6.1)$$

The relationship between the asymmetric quantum correlator  $\langle \hat{V}(\mathbf{r}_2, t) \hat{V}(\mathbf{r}_1, \tau) \rangle$  and the symmetric correlation function

$$\Gamma_s(\mathbf{r}_1 - \mathbf{r}_2, t - \tau) = \frac{1}{2} (\langle \hat{V}(\mathbf{r}_2, t) \hat{V}(\mathbf{r}_1, \tau) \rangle + \langle \hat{V}(\mathbf{r}_1, \tau) \hat{V}(\mathbf{r}_2, t) \rangle) \quad (6.2)$$

is given by the fluctuation-dissipation theorem (equation (4.6)):

$$\begin{aligned} \langle \hat{V}(\mathbf{r}_2, t) \hat{V}(\mathbf{r}_1, \tau) \rangle &= 2(1 + \exp(-i\beta\hbar\partial/\partial t))^{-1} \Gamma_s(\mathbf{r}_1 - \mathbf{r}_2, t - \tau) \\ &\approx (1 + \frac{1}{2}i\beta\hbar\partial/\partial t) \Gamma_s(\mathbf{r}_1 - \mathbf{r}_2, t - \tau) \end{aligned} \quad (6.3)$$

or, analogously,

$$\langle \hat{V}(\mathbf{r}_1, \tau) \hat{V}(\mathbf{r}_2, t) \rangle \approx (1 - \frac{1}{2}i\beta\hbar\partial/\partial t) \Gamma_s(\mathbf{r}_1 - \mathbf{r}_2, t - \tau). \quad (6.4)$$

The next simplifications can be obtained if the correlation length  $l_{\text{corr}}$  of the correlator  $\Gamma_s(\mathbf{r}_1 - \mathbf{r}_2, t - \tau)$  is much more than the length of localisation  $l_{\text{loc}}$  of a wavepacket,  $l_{\text{loc}}/l_{\text{corr}} \ll 1$ . Then all field correlators may be expanded in terms of  $(\mathbf{r}_1 - \mathbf{r}_2)$  by using the parameter  $l_{\text{loc}}/l_{\text{corr}}$ :

$$\begin{aligned} & \langle \hat{V}(\mathbf{r}_2, t) \hat{V}(\mathbf{r}_1, \tau) \rangle \\ & \approx \langle \hat{V}(\mathbf{r}_1, t) \hat{V}(\mathbf{r}_1, \tau) \rangle + (r_{2j} - r_{1j}) \left\langle \frac{\partial \hat{V}(\mathbf{r}_1, t)}{\partial r_{1j}} \hat{V}(\mathbf{r}_1, \tau) \right\rangle \\ & \quad + \frac{1}{2} (r_{2j} - r_{1j})(r_{2k} - r_{1k}) \left\langle \frac{\partial^2 \hat{V}(\mathbf{r}_1, t)}{\partial r_{1j} \partial r_{1k}} \hat{V}(\mathbf{r}_1, \tau) \right\rangle. \end{aligned} \quad (6.5)$$

Here summation is performed over all repeated indices. In the isotropic case one obtains

$$\left\langle \frac{\partial \hat{V}(\mathbf{r}_1, t)}{\partial r_{1j}} \hat{V}(\mathbf{r}_1, \tau) \right\rangle = 0 \quad (6.6)$$

$$\begin{aligned} \left\langle \frac{\partial \hat{V}(\mathbf{r}_1, t)}{\partial r_{1j}} \frac{\partial \hat{V}(\mathbf{r}_1, \tau)}{\partial r_{1k}} \right\rangle &= - \left\langle \frac{\partial^2 \hat{V}(\mathbf{r}_1, t)}{\partial r_{1j} \partial r_{1k}} \hat{V}(\mathbf{r}_1, \tau) \right\rangle \\ &= \frac{1}{d} \delta_{jk} \left\langle \frac{\partial \hat{V}(\mathbf{r}_1, t)}{\partial r_{1n}} \frac{\partial \hat{V}(\mathbf{r}_1, \tau)}{\partial r_{1n}} \right\rangle \end{aligned} \quad (6.7)$$

where  $d$  is the dimensionality of space.

Collecting all these equalities, the final result can be written after lengthy but straightforward algebra in the form

$$\begin{aligned} & \frac{\partial}{\partial t} \langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle \\ & = -\frac{i}{\hbar} (\hat{H}(\mathbf{r}_1) - \hat{H}(\mathbf{r}_2) + \hat{H}'(\mathbf{r}_1, \mathbf{r}_2)) \langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle \\ & \quad - \frac{\gamma}{m} (\mathbf{r}_2 - \mathbf{r}_1) \left( \frac{\partial}{\partial \mathbf{r}_2} - \frac{\partial}{\partial \mathbf{r}_1} \right) \langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle \\ & \quad - \frac{D^{(p)}}{2\hbar^2} |\mathbf{r}_1 - \mathbf{r}_2|^2 \langle \psi^*(\mathbf{r}_2, t) \psi(\mathbf{r}_1, t) \rangle \end{aligned} \quad (6.8)$$

where

$$D^{(p)} = \frac{1}{d} \int_0^\infty d\tau \left( \left\langle \frac{\partial \hat{V}(\mathbf{r}, \tau)}{\partial \mathbf{r}} \frac{\partial \hat{V}(\mathbf{r}, t)}{\partial \mathbf{r}} \right\rangle + \left\langle \frac{\partial \hat{V}(\mathbf{r}, t)}{\partial \mathbf{r}} \frac{\partial \hat{V}(\mathbf{r}, \tau)}{\partial \mathbf{r}} \right\rangle \right) \tag{6.9}$$

$$\gamma = \beta D^{(p)} / 2 \tag{6.10}$$

$$\hat{H}'(\mathbf{r}_1, \mathbf{r}_2) = \frac{\zeta}{m} (\mathbf{r}_1 - \mathbf{r}_2) \left( \frac{\partial}{\partial \mathbf{r}_1} + \frac{\partial}{\partial \mathbf{r}_2} \right) \tag{6.11}$$

$$\zeta = \frac{1}{2d} \lim_{t \rightarrow \infty} \int_0^t d\tau (t - \tau) \left( \left\langle \frac{\partial \hat{V}(\mathbf{r}, t)}{\partial \mathbf{r}} \frac{\partial \hat{V}(\mathbf{r}, \tau)}{\partial \mathbf{r}} \right\rangle + \left\langle \frac{\partial \hat{V}(\mathbf{r}, \tau)}{\partial \mathbf{r}} \frac{\partial \hat{V}(\mathbf{r}, t)}{\partial \mathbf{r}} \right\rangle \right). \tag{6.12}$$

Equation (6.8) coincides, up to the slight difference due to the term  $\hat{H}'(\mathbf{r}_1, \mathbf{r}_2)$ , with the equations derived previously by Dekker (1977) and Caldeira and Leggett (1983a). There is, however, a certain difference in these correspondences. As has been pointed out in § 1, there is no principal non-equivalence between the Schrödinger equation and path-integral formulations. Hence, the correspondence between our result and that of Caldeira and Leggett (1983a) is obligatory. In both cases the equations are linear. On the other hand, Dekker (1977) has used a non-linear formalism and obtained an analogous equation within certain approximations of non-linear theory. It has been shown in these works that quasiclassical correspondence can be most naturally stated after transition to the Wigner distribution function:

$$f_w(\mathbf{R}, \mathbf{p}, t) = \frac{1}{(2\pi\hbar)^d} \int d^d \boldsymbol{\rho} \exp(-i\mathbf{p}\boldsymbol{\rho}/\hbar) \times \langle \psi^*(\mathbf{R} - \boldsymbol{\rho}/2, t) \psi(\mathbf{R} + \boldsymbol{\rho}/2, t) \rangle. \tag{6.13}$$

Equation (6.8) then transforms to

$$\begin{aligned} \frac{\partial f_w}{\partial t} = & -\frac{1}{m} \frac{\partial}{\partial \mathbf{R}} (\mathbf{p} f_w) - \frac{i}{\hbar} \left[ U\left(\mathbf{R} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}}\right) - U\left(\mathbf{R} - \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}}\right) \right] f_w \\ & + \frac{\gamma}{m} \frac{\partial}{\partial \mathbf{p}} (\mathbf{p} f_w) + \frac{\zeta}{m} \frac{\partial^2 f_w}{\partial \mathbf{R} \partial \mathbf{p}} + \frac{1}{2} D^{(p)} \frac{\partial^2 f_w}{\partial \mathbf{p}^2}. \end{aligned} \tag{6.14}$$

In the limit  $\hbar \rightarrow 0$  one can expand

$$U\left(\mathbf{R} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}}\right) \approx U(\mathbf{R}) + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}} \frac{\partial U(\mathbf{R})}{\partial \mathbf{R}} \tag{6.15}$$

and use  $\zeta \sim D^{(p)} \beta \hbar \rightarrow 0$  (see equation (6.12)). As can be seen, in this limit equation (6.14) is reduced to the classical Fokker-Planck equation, while equation (6.10) corresponds to the classical Einstein relationship.

Equation (6.8) can be applied to various quasiclassical problems. For example, for the free particle ( $U(\mathbf{r}) = 0$ ) it is easy to obtain the analogue of Ehrenfest's equations (see, e.g., Landau and Lifshitz 1976):

$$\frac{\partial \langle \mathbf{R}^2 \rangle}{\partial t} = \langle \hat{\mathbf{v}} \mathbf{R} + \mathbf{R} \hat{\mathbf{v}} \rangle \tag{6.16}$$

$$\frac{\partial \langle \hat{\mathbf{v}} \mathbf{R} + \mathbf{R} \hat{\mathbf{v}} \rangle}{\partial t} = 2 \langle \hat{\mathbf{v}}^2 \rangle + \frac{2d\zeta}{m} - \frac{\gamma}{m} \langle \hat{\mathbf{v}} \mathbf{R} + \mathbf{R} \hat{\mathbf{v}} \rangle \tag{6.17}$$

$$\frac{\partial \langle \hat{\mathbf{v}}^2 \rangle}{\partial t} = -\frac{2\gamma}{m} \langle \hat{\mathbf{v}}^2 \rangle + \frac{dD^{(p)}}{m} \tag{6.18}$$

where  $\hat{v} = (-i\hbar/m) \partial/\partial \mathbf{r}$  is the velocity operator. The asymptotic evolution at large times is given by

$$\langle \mathbf{R}^2 \rangle \approx \frac{2d}{\gamma\beta} \left( 1 + \frac{\zeta\beta}{m} \right) t. \quad (6.19)$$

Comparing this result with the expression for classical diffusion

$$\langle \mathbf{R}^2 \rangle = 2dD^{(R)}t \quad (6.20)$$

we obtain the correspondence

$$D^{(R)} = \frac{1}{\gamma\beta} \left( 1 + \frac{\zeta\beta}{m} \right) \quad (6.21)$$

which, up to slight quantum corrections, coincides with classical theory.

It is evident from the derivation of equation (6.8) that dissipation terms are due to the first non-vanishing corrections to the parameter  $\langle \hat{H}(\mathbf{r}) \rangle \tau_{\text{corr}}/\hbar$  and will be equal to zero within the pure white-noise approximation. For this reason, equation (2.41) does not contain any dissipation effects. Acting on both sides of this equation by  $\hat{H}(\mathbf{r}_1)$ , tending  $\mathbf{r}_2$  to  $\mathbf{r}_1$  and integrating over  $\mathbf{r}_1$  one obtains

$$\frac{\partial \langle \hat{H}(\mathbf{r}) \rangle}{\partial t} = \frac{dD^{(P)}}{2m} \quad (6.22)$$

which corresponds to stochastic pumping of energy only. The mean squared distance  $\langle \mathbf{R}^2 \rangle$  will in this case behave asymptotically according to the cubic law  $\langle \mathbf{R}^2 \rangle \propto t^3$  (Jayannavar and Kumar 1982) rather than  $\langle \mathbf{R}^2 \rangle \propto t$ .

Equation (6.8) can also be applied to tunnelling problems at high temperatures. Within such an approximation this problem has been studied by Büttiker *et al* (1983), Hänggi and Weiss (1984) and Mel'nikov (1985) (see also the review by Hänggi 1986).

Finally, we should stress that the use of the quasiclassical Fokker-Planck approximation depends physically on the presence of low-lying modes ensuring small transfers of energy and momentum (cf § 3). The last condition can be fulfilled often only within a certain range of temperatures and may be violated, e.g., at very low (or very high) temperatures. Therefore, attempts to obtain exact quantum expressions for the diffusion coefficient universally valid at all temperatures must be made with care.

## 7. Multiparticle systems

We shall show in this section how all results can be generalised to multiparticle systems. The corresponding Schrödinger equation is written in the form

$$i\hbar \frac{\partial \psi^{(K)}}{\partial t} = \hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K) \psi^{(K)} + V^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \psi^{(K)} \quad (7.1)$$

where

$$\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K) = \sum_{\alpha=1}^K -\frac{\hbar^2}{2m_\alpha} \frac{\partial^2}{\partial \mathbf{r}_\alpha^2} + U(\mathbf{r}_1, \dots, \mathbf{r}_K) \quad (7.2)$$

$$V^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) = \sum_{\alpha=1}^K V(\mathbf{r}_\alpha, t). \quad (7.3)$$

All  $K$  particles are taken for simplicity to be non-identical, while quantum effects related to identity and statistics will be discussed separately below. The correlations of the random potential  $V^{(K)}(\mathbf{r}_\alpha; t)$  are expressed in terms of that of  $V(\mathbf{r}_\alpha, t)$  (see equations (2.2) and (2.3)):

$$\langle V^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \rangle = 0 \tag{7.4}$$

$$\begin{aligned} \langle V^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; t') V^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \rangle \\ = \left\langle \left( \sum_\alpha V(\mathbf{r}'_\alpha, t') \right) \left( \sum_\beta V(\mathbf{r}_\beta, t) \right) \right\rangle = \sum_{\alpha, \beta} \Gamma(\mathbf{r}'_\alpha - \mathbf{r}_\beta, t' - t). \end{aligned} \tag{7.5}$$

We give them first for  $c$ -number quantities. As the sum of Gaussian quantities again remains Gaussian (see, e.g., Feller 1970), it is evident that all generalisations required consist of a mere expansion of configurational space, while the main features of the derivation given in § 2 will be essentially the same. For example, the evolution equation for the averaged  $\psi^{(K)}$  function is given by (cf equation (2.16))

$$\begin{aligned} \frac{\partial \langle \psi^{(K)} \rangle}{\partial t} = -\frac{i}{\hbar} \hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K) \langle \psi^{(K)} \rangle \\ - \frac{1}{\hbar^2} \int_0^t d\tau \langle V^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \exp(-i\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K)(t-\tau)/\hbar) \\ \times V^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; \tau) \exp(i\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K)(t-\tau)/\hbar) \rangle \langle \psi^{(K)} \rangle. \end{aligned} \tag{7.6}$$

As is seen from equations (7.2)-(7.6), if the correlation length of the potentials  $V(\mathbf{r}_\alpha, t)$  is much more than the length of localisation of the  $K$ -particle system, then interference effects will dominate in equation (7.6) and the corresponding dephasing time will be proportional to  $K^2$ , while in the opposite limit the stochastisation will be incoherent and the dephasing time will be proportional to  $K$ .

The evolution of the density matrix is determined by the equation (cf equation (2.32))

$$\begin{aligned} \frac{\partial}{\partial t} \langle \psi^{*(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; t) \psi^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \rangle \\ = -\frac{i}{\hbar} (\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K) - \hat{H}^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K)) \\ \times \langle \psi^{*(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; t) \psi^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \rangle \\ - \frac{1}{\hbar^2} \int_0^t d\tau [ \langle V^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \exp(-i\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K)(t-\tau)/\hbar) \\ \times V^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; \tau) \exp(i\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K)(t-\tau)/\hbar) \\ + \langle V^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; t) \exp(i\hat{H}^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K)(t-\tau)/\hbar) \\ \times V^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; \tau) \exp(-i\hat{H}^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K)(t-\tau)/\hbar) \\ - \langle V^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \exp(i\hat{H}^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K)(t-\tau)/\hbar) \\ \times V^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; \tau) \exp(-i\hat{H}^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K)(t-\tau)/\hbar) \\ - \langle V^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; t) \exp(-i\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K)(t-\tau)/\hbar) \\ \times V^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; \tau) \exp(i\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K)(t-\tau)/\hbar) ] \\ \times \langle \psi^{*(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; t) \psi^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \rangle. \end{aligned} \tag{7.7}$$

For the more general case of quantised fields  $\hat{V}(\mathbf{r}_\alpha, t)$  all operators

$$\langle \hat{V}^{(K)}(\mathbf{r}; t) \exp(i\hat{H}^{(K)}(\mathbf{r}')(t-\tau)/\hbar) \hat{V}^{(K)}(\mathbf{r}', \tau) \exp(-i\hat{H}^{(K)}(\mathbf{r}')(t-\tau)/\hbar) \rangle$$

acting from left to right should be replaced by the operators

$$\langle \exp(-i\hat{H}^{(K)}(\mathbf{r}')(t-\tau)/\hbar) \hat{V}^{(K)}(\mathbf{r}', \tau) \exp(i\hat{H}^{(K)}(\mathbf{r}')(t-\tau)/\hbar) \hat{V}(\mathbf{r}, t) \rangle$$

acting from right to left.

In the high-temperature limit, equation (7.7) can be approximated by the  $K$ -particle quasiclassical Fokker-Planck equation if the correlation time of field fluctuations is much less than the reciprocal characteristic eigenfrequencies of the Hamiltonian  $\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K)$  and the corresponding correlation length is much greater than the characteristic length of localisation of the  $K$ -particle system:

$$\begin{aligned} & \frac{\partial}{\partial t} \langle \psi^{*(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; t) \psi^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \rangle \\ &= -\frac{i}{\hbar} (\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K) - \hat{H}^{(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K)) \\ & \quad + \hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; \mathbf{r}'_1, \dots, \mathbf{r}'_K) \\ & \quad \times \langle \psi^{*(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; t) \psi^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \rangle \\ & \quad - \frac{1}{2} \gamma \sum_{\alpha, \beta} \left[ (\mathbf{r}'_\alpha - \mathbf{r}_\beta) \left( \frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{r}'_\alpha} - \frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{r}_\beta} \right) \right. \\ & \quad - \frac{1}{2} (\mathbf{r}_\alpha - \mathbf{r}_\beta) \left( \frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{r}_\alpha} - \frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{r}_\beta} \right) \\ & \quad \left. - \frac{1}{2} (\mathbf{r}'_\alpha - \mathbf{r}'_\beta) \left( \frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{r}'_\alpha} - \frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{r}'_\beta} \right) \right] \\ & \quad \times \langle \psi^{*(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; t) \psi^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \rangle \\ & \quad - \frac{D^{(p)}}{2\hbar^2} \sum_{\alpha, \beta} (|\mathbf{r}'_\alpha - \mathbf{r}_\beta|^2 - \frac{1}{2} |\mathbf{r}_\alpha - \mathbf{r}_\beta|^2 - \frac{1}{2} |\mathbf{r}'_\alpha - \mathbf{r}'_\beta|^2) \\ & \quad \times \langle \psi^{*(K)}(\mathbf{r}'_1, \dots, \mathbf{r}'_K; t) \psi^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; t) \rangle \end{aligned} \quad (7.8)$$

where

$$\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K; \mathbf{r}'_1, \dots, \mathbf{r}'_K) = \zeta \sum_{\alpha, \beta} (\mathbf{r}_\alpha - \mathbf{r}'_\beta) \left( \frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{r}_\alpha} + \frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{r}'_\beta} \right). \quad (7.9)$$

The coefficients  $D^{(p)}$ ,  $\gamma$  and  $\zeta$  are defined by equations (6.9), (6.10) and (6.12).

Equations (7.7)–(7.9) also retain their formal structure for a system of identical particles (see, e.g., Feynman 1972). The effects of quantum statistics (Bose or Fermi) must be taken into account by imposing additional external restrictions on the symmetry properties of the various correlators with respect to mutual permutations of coordinates of identical particles. For Bose statistics they must be symmetrical under this operation, while for Fermi particles they must be antisymmetrised. The statistics also restricts the structure of corresponding expansions using eigenfunctions of the Hamiltonian  $\hat{H}^{(K)}(\mathbf{r}_1, \dots, \mathbf{r}_K)$  (cf equation (4.1)). For example, in the case of Fermi statistics they must be taken in the form of Slater determinants, etc. It is evident, however, that proof given in § 4 remains essentially unchanged.

### 8. Stochastic evolution in the space of $\psi$ functions

#### 8.1. Continuous Fokker-Planck equation

The stochastic evolution of a quantum system under action of a random potential can also be described with the use of an alternative approach. From the point of view of the complete set of states  $\{\psi(\mathbf{r})\}$  (we shall consider for simplicity the one-particle problem only) the subsequent stochastic evolution of a given initial state  $\psi_0(\mathbf{r})$  is equivalent to continuous diffusion in the space of  $\psi$  functions. Therefore the problem may be reformulated in terms of the continuous Fokker-Planck equation in the space of  $\psi$  functions. We first describe this equation using the more simple white-noise approximation (2.39).

The corresponding equation is derived by continuous generalisation of the standard mathematical technique for multivariable problems with multiplicative noise (see, e.g., Kljatskin (1975) and appendix 1) and has the form

$$\begin{aligned} \frac{\partial \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\}}{\partial t} &= \hat{L}\mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\} \\ &+ \frac{1}{2} \iint d^d \mathbf{r} d^d \mathbf{r}' \Gamma^{(0)}(\mathbf{r} - \mathbf{r}') \hat{D}(\mathbf{r}) \hat{D}(\mathbf{r}') \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\} \end{aligned} \tag{8.1}$$

where

$$\hat{L} = \frac{i}{\hbar} \int d^d \mathbf{r} \left( \frac{\delta}{\delta \psi(\mathbf{r})} \hat{H}(\mathbf{r}) \psi(\mathbf{r}) - \frac{\delta}{\delta \psi^*(\mathbf{r})} \hat{H}(\mathbf{r}) \psi^*(\mathbf{r}) \right) \tag{8.2}$$

$$\hat{D}(\mathbf{r}) = -\frac{i}{\hbar} \left( \frac{\delta}{\delta \psi(\mathbf{r})} \psi(\mathbf{r}) - \frac{\delta}{\delta \psi^*(\mathbf{r})} \psi^*(\mathbf{r}) \right) \tag{8.3}$$

and  $\delta/\delta\psi(\mathbf{r})$ ,  $\delta/\delta\psi^*(\mathbf{r})$  are functional derivatives. The action of the operator  $(\delta/\delta\psi(\mathbf{r}))\psi(\mathbf{r})$  (and correspondingly  $(\delta/\delta\psi^*(\mathbf{r}))\psi^*(\mathbf{r})$ ) is equivalent to multiplication by a function  $\psi(\mathbf{r})$  and subsequent functional differentiation. We used  $\psi(\mathbf{r})$  and the complex conjugate  $\psi^*(\mathbf{r})$  as independent variables instead of the less convenient  $\text{Re } \psi(\mathbf{r})$  and  $\text{Im } \psi(\mathbf{r})$ . The functional  $\mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\}$  describes the probability of finding a state  $\psi(\mathbf{r})$  at a moment  $t$  and is normalised to unity:

$$\int \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\} \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) = 1. \tag{8.4}$$

Here the integration is understood in the functional sense (see, e.g., Feynman and Hibbs 1965, Vasil'ev 1976, Itzykson and Zuber 1980).

All equal time averages are equal by definition to

$$\begin{aligned} \langle \psi(\mathbf{r}_1, t) \dots \psi^*(\mathbf{r}_n, t) \rangle &\equiv \langle \psi(\mathbf{r}_1) \dots \psi^*(\mathbf{r}_n); t \rangle \\ &= \int \psi(\mathbf{r}_1) \dots \psi^*(\mathbf{r}_n) \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\} \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}). \end{aligned} \tag{8.5}$$

The corresponding evolution equations for averages (8.5) can be derived with the use of the trick introduced first by Feynman in quantum field theory (Feynman and Hibbs



(1965); similar methods have been used by Zubarev and Morozov (1983) in the study of hydrodynamic fluctuations and by Chechetkin and Lutovinov (1986) in the continuous stochastic theory of birth and death processes). Replacing the variables  $\psi(\mathbf{r})$  and  $\psi^*(\mathbf{r})$  by

$$\psi(\mathbf{r}) = \tilde{\psi}(\mathbf{r}) + \eta(\mathbf{r}) \quad \psi^*(\mathbf{r}) = \tilde{\psi}^*(\mathbf{r}) + \eta^*(\mathbf{r}) \quad (8.6)$$

shifted on the fixed arbitrary functions  $\eta(\mathbf{r})$  and  $\eta^*(\mathbf{r})$  one obtains, after expansion of the normalisation condition (8.4) into a functional Taylor series, the following set of equalities:

$$\begin{aligned} & \int \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\} \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) \\ &= \int \mathcal{P}\{\tilde{\psi}(\mathbf{r}) + \eta(\mathbf{r}), \tilde{\psi}^*(\mathbf{r}) + \eta^*(\mathbf{r}); t\} \mathcal{D}\tilde{\psi}(\mathbf{r}) \mathcal{D}\tilde{\psi}^*(\mathbf{r}) \\ &= \int \mathcal{P}\{\tilde{\psi}(\mathbf{r}), \tilde{\psi}^*(\mathbf{r}); t\} \mathcal{D}\tilde{\psi}(\mathbf{r}) \mathcal{D}\tilde{\psi}^*(\mathbf{r}) \\ &+ \int \int \frac{\delta \mathcal{P}\{\tilde{\psi}(\mathbf{r}), \tilde{\psi}^*(\mathbf{r}); t\}}{\delta \psi(\mathbf{r}')} \eta(\mathbf{r}') d^d \mathbf{r}' \mathcal{D}\tilde{\psi}(\mathbf{r}) \mathcal{D}\tilde{\psi}^*(\mathbf{r}) \\ &+ \int \int \frac{\delta \mathcal{P}\{\tilde{\psi}(\mathbf{r}), \tilde{\psi}^*(\mathbf{r}); t\}}{\delta \psi^*(\mathbf{r}')} \eta^*(\mathbf{r}') d^d \mathbf{r}' \mathcal{D}\tilde{\psi}(\mathbf{r}) \mathcal{D}\tilde{\psi}^*(\mathbf{r}) + \dots \quad (8.7) \end{aligned}$$

In view of the arbitrariness of the functions  $\eta(\mathbf{r})$  and  $\eta^*(\mathbf{r})$  this equality can be satisfied only if

$$\begin{aligned} & \int \frac{\delta \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\}}{\delta \psi(\mathbf{r}')} \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) \\ &= \int \frac{\delta \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\}}{\delta \psi^*(\mathbf{r}')} \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) \\ &= \int \frac{\delta^2 \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\}}{\delta \psi(\mathbf{r}') \delta \psi(\mathbf{r}'')} \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) = \dots = 0. \quad (8.8) \end{aligned}$$

Similar equalities are valid for any functional  $\Phi\{\psi(\mathbf{r}), \psi^*(\mathbf{r})\}$  which does not grow too rapidly at infinity:

$$\int \frac{\delta}{\delta \psi(\mathbf{r}')} (\Phi\{\psi(\mathbf{r}), \psi^*(\mathbf{r})\} \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\}) \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) = \dots = 0. \quad (8.9)$$

This allows the use of simple functional integration by parts, i.e.

$$\int \psi(\mathbf{r}'') \frac{\delta \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\}}{\delta \psi(\mathbf{r}')} \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) = -\delta(\mathbf{r}' - \mathbf{r}''). \quad (8.10)$$

We have used in the integration the normalisation condition (8.4) and the equalities

$$\frac{\delta \psi(\mathbf{r})}{\delta \psi(\mathbf{r}')} = \frac{\delta \psi^*(\mathbf{r})}{\delta \psi^*(\mathbf{r}')} = \delta(\mathbf{r} - \mathbf{r}') \quad (8.11)$$

$$\frac{\delta \psi(\mathbf{r})}{\delta \psi^*(\mathbf{r}')} = \frac{\delta \psi^*(\mathbf{r})}{\delta \psi(\mathbf{r}')} = 0. \quad (8.12)$$

Thus evolution equations for the averages (8.5) are obtained by (i) multiplication of both sides of equation (8.1) by  $\psi(\mathbf{r}_1) \dots \psi^*(\mathbf{r}_n)$ ; (ii) functional integration over  $\psi(\mathbf{r})$  and  $\psi^*(\mathbf{r})$ ; and (iii) subsequent functional integration by parts with the use of the equalities (8.4), (8.9), (8.11) and (8.12). For example, one obtains (see equations (8.1)–(8.3))

$$\begin{aligned} \frac{\partial}{\partial t} \left( \int \psi(\tilde{\mathbf{r}}) \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\} \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) \right) &= \frac{\partial \langle \psi(\tilde{\mathbf{r}}, t) \rangle}{\partial t} \\ &= \int \psi(\mathbf{r}) \left( \hat{L} \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\} \right. \\ &\quad \left. + \frac{1}{2} \iint d^d \mathbf{r} d^d \mathbf{r}' \Gamma^{(0)}(\mathbf{r} - \mathbf{r}') \hat{D}(\mathbf{r}) \hat{D}(\mathbf{r}') \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\} \right) \\ &\quad \times \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) \\ &= -\frac{i}{\hbar} \hat{H}(\tilde{\mathbf{r}}) \langle \psi(\tilde{\mathbf{r}}, t) \rangle - \frac{\Gamma^{(0)}(0)}{2\hbar^2} \langle \psi(\tilde{\mathbf{r}}, t) \rangle \end{aligned} \tag{8.13}$$

which coincides with equation (2.40).

In the general case it is necessary to use a generalised continuous Fokker–Planck equation of the form

$$\begin{aligned} \frac{\partial \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\}}{\partial t} &= \hat{L} \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\} \\ &\quad + \iint d^d \mathbf{r} d^d \mathbf{r}' \int_0^t d\tau \hat{\Gamma}(\mathbf{r}, t; \mathbf{r}', \tau) \hat{D}(\mathbf{r}) \hat{D}(\mathbf{r}') \mathcal{P}\{\psi(\mathbf{r}), \psi^*(\mathbf{r}); t\} \end{aligned} \tag{8.14}$$

where the operators  $\hat{L}$  and  $\hat{D}(\mathbf{r})$  have been defined in equations (8.2) and (8.3), and  $\hat{\Gamma}(\mathbf{r}, t; \mathbf{r}', \tau) = \langle V(\mathbf{r}, t) \exp(-i\varepsilon(\psi) \hat{H}(\mathbf{r}')(t - \tau) / \hbar) V(\mathbf{r}', \tau) \exp(i\varepsilon(\psi) \hat{H}(\mathbf{r}')(t - \tau) / \hbar) \rangle$ . (8.15)

The operator  $\hat{\Gamma}(\mathbf{r}, t; \mathbf{r}', \tau)$  must always be posed to the extreme left position,  $\varepsilon(\psi)$  is equal to +1 if the coordinate  $\mathbf{r}'$  corresponds to that of  $\psi(\mathbf{r}')$  for the average  $\langle \dots \psi(\mathbf{r}') \dots \rangle$ , and  $\varepsilon(\psi)$  is equal to -1 if the coordinate  $\mathbf{r}'$  corresponds to that of  $\psi^*(\mathbf{r}')$  for the average  $\langle \dots \psi^*(\mathbf{r}') \dots \rangle$ . In the more exact form for quantised fields  $\hat{V}(\mathbf{r}, t)$ , the order of action should be changed from left to right in the last case (see discussion at the end of § 2).

### 8.2. Thermoequilibrium distribution in the space of $\psi$ functions

The thermoequilibrium distribution in the space of  $\psi$  functions can be most conveniently derived with the use of the characteristic functional

$$\Phi_T^{(\psi)}\{\lambda(\mathbf{r}), \lambda^*(\mathbf{r})\} = \left\langle \exp \left( i \int d^d \mathbf{r} (\lambda^*(\mathbf{r}) \psi^*(\mathbf{r}) + \lambda(\mathbf{r}) \psi(\mathbf{r})) \right) \right\rangle. \tag{8.16}$$

The corresponding correlators are obtained by the functional differentiation

$$\langle \psi^*(\mathbf{r}_n) \dots \psi(\mathbf{r}_1) \rangle_T = \frac{\delta \dots \Phi_T^{(\psi)}\{\lambda(\mathbf{r}), \lambda^*(\mathbf{r})\}}{i\delta\lambda^*(\mathbf{r}_n) \dots i\delta\lambda(\mathbf{r}_1)} \Bigg|_{\lambda(\mathbf{r})=\lambda^*(\mathbf{r})=0} \tag{8.17}$$

In thermoequilibrium only correlators with equal numbers of  $\psi(\mathbf{r})$  and  $\psi^*(\mathbf{r})$  will be not equal to zero due to phase fluctuations (cf Landau and Lifshitz 1969). For this reason the characteristic functional  $\Phi_T^{(\psi)}\{\lambda(\mathbf{r}), \lambda^*(\mathbf{r})\}$  may be written in the form

$$\begin{aligned} \ln \Phi_T^{(\psi)}\{\lambda(\mathbf{r}), \lambda^*(\mathbf{r})\} &= - \sum_{n=1}^{\infty} \frac{1}{(n!)^2} \int \dots \int d^d\mathbf{r}'_1 \dots d^d\mathbf{r}'_n d^d\mathbf{r}_1 \dots d^d\mathbf{r}_n \lambda^*(\mathbf{r}'_1) \dots \lambda^*(\mathbf{r}'_n) \\ &\quad \times Q_n(\mathbf{r}'_1, \dots, \mathbf{r}'_n; \mathbf{r}_1, \dots, \mathbf{r}_n) \lambda(\mathbf{r}_1) \dots \lambda(\mathbf{r}_n). \end{aligned} \tag{8.18}$$

Here the cumulant functions  $\{Q_n(\mathbf{r}'_1, \dots, \mathbf{r}'_n; \mathbf{r}_1, \dots, \mathbf{r}_n)\}$  are symmetric under various permutations of the coordinates  $\{\mathbf{r}_\alpha\}$  and  $\{\mathbf{r}'_\alpha\}$ .

The function  $Q_1(\mathbf{r}', \mathbf{r})$  is equal by definition (8.17) to the thermodynamic density matrix (see equations (4.10) and (4.11))

$$Q_1(\mathbf{r}', \mathbf{r}) = \langle \psi^*(\mathbf{r}')\psi(\mathbf{r}) \rangle_T. \tag{8.19}$$

The structure of the other functions is determined by induction with the use of the reduction equality (4.14) which is equivalent to the exact conservation of normalisation and corresponding differentiation of equation (8.18). For example,

$$\begin{aligned} \frac{\delta^4 \ln \Phi_T^{(\psi)}\{\lambda(\mathbf{r}), \lambda^*(\mathbf{r})\}}{i^4 \delta\lambda^*(\mathbf{r}'_2) \delta\lambda^*(\mathbf{r}'_1) \delta\lambda(\mathbf{r}_2) \delta\lambda(\mathbf{r}_1)} \Bigg|_{\lambda(\mathbf{r})=\lambda^*(\mathbf{r})=0} &= \langle \psi^*(\mathbf{r}'_2)\psi^*(\mathbf{r}'_1)\psi(\mathbf{r}_2)\psi(\mathbf{r}_1) \rangle_T - \langle \psi^*(\mathbf{r}'_2)\psi(\mathbf{r}_2) \rangle_T \langle \psi^*(\mathbf{r}'_1)\psi(\mathbf{r}_1) \rangle_T \\ &\quad - \langle \psi^*(\mathbf{r}'_2)\psi(\mathbf{r}_1) \rangle_T \langle \psi^*(\mathbf{r}'_1)\psi(\mathbf{r}_2) \rangle_T \\ &= -Q_4(\mathbf{r}'_2, \mathbf{r}'_1; \mathbf{r}_2, \mathbf{r}_1). \end{aligned} \tag{8.20}$$

Then the reduction equality (4.14) gives

$$Q_4(\mathbf{r}'_2, \mathbf{r}'_1; \mathbf{r}_2, \mathbf{r}_1) = Z^{-2} \sum_j e^{-2\beta E_j} \psi_j^*(\mathbf{r}'_2)\psi_j^*(\mathbf{r}'_1)\psi_j(\mathbf{r}_2)\psi_j(\mathbf{r}_1). \tag{8.21}$$

Analogously, one can obtain

$$\begin{aligned} Q_6(\mathbf{r}'_3, \mathbf{r}'_2, \mathbf{r}'_1; \mathbf{r}_3, \mathbf{r}_2, \mathbf{r}_1) &= 4Z^{-3} \sum_j e^{-3\beta E_j} \psi_j^*(\mathbf{r}'_3)\psi_j^*(\mathbf{r}'_2)\psi_j^*(\mathbf{r}'_1)\psi_j(\mathbf{r}_3)\psi_j(\mathbf{r}_2)\psi_j(\mathbf{r}_1). \end{aligned} \tag{8.22}$$

It can be proved (the argument is analogous to the Mayer theorem and is based on the theorem on the connectedness of the diagrammatic representation of the logarithm (see, e.g., Abrikosov *et al* 1963, Vasil'ev 1976)) that the general structure of the functions  $Q_n(\mathbf{r}'_n, \dots, \mathbf{r}'_1; \mathbf{r}_n, \dots, \mathbf{r}_1)$  is given by

$$\begin{aligned} Q_n(\mathbf{r}'_n, \dots, \mathbf{r}'_1; \mathbf{r}_n, \dots, \mathbf{r}_1) &= \gamma_n Z^{-n} \sum_j e^{-n\beta E_j} \psi_j^*(\mathbf{r}'_n) \dots \psi_j^*(\mathbf{r}'_1)\psi_j(\mathbf{r}_n) \dots \psi_j(\mathbf{r}_1) \end{aligned} \tag{8.23}$$

where  $\{\gamma_n\}$  are some combinatoric coefficients.

Using standard methods (see, e.g., Vasil'ev 1976, Slavnov and Faddeev 1978, Itzykson and Zuber 1980 and appendix 2) the reciprocal Fourier transform can be written in the form

$$\begin{aligned} \mathcal{P}_T\{\psi(\mathbf{r}), \psi^*(\mathbf{r})\} &= N \exp\left(-\hat{Q}\left\{\frac{\delta}{i\delta\eta(\mathbf{r})}, \frac{\delta}{i\delta\eta^*(\mathbf{r})}\right\}\right) \\ &\times \exp\left(-Z \int d^d\mathbf{r}(\psi^*(\mathbf{r}) + \eta^*(\mathbf{r})) e^{\beta\hat{H}(\mathbf{r})}(\psi(\mathbf{r}) + \eta(\mathbf{r}))\right) \Big|_{\eta(\mathbf{r})=\eta^*(\mathbf{r})=0} \end{aligned} \quad (8.24)$$

where  $N$  is a normalisation coefficient,  $\hat{H}(\mathbf{r})$  has been defined in equation (2.1), and

$$\begin{aligned} \hat{Q}\left\{\frac{\delta}{i\delta\eta(\mathbf{r})}, \frac{\delta}{i\delta\eta^*(\mathbf{r})}\right\} &= \sum_{n=2}^{\infty} \int \dots \int d^d\mathbf{r}'_n \dots d^d\mathbf{r}'_1 d^d\mathbf{r}_n \dots d^d\mathbf{r}_1 \\ &\times \frac{1}{(n!)^2} Q_n(\mathbf{r}'_n, \dots, \mathbf{r}'_1; \mathbf{r}_n, \dots, \mathbf{r}_1) \\ &\times \frac{\delta}{i\delta\eta^*(\mathbf{r}'_n)} \dots \frac{\delta}{i\delta\eta^*(\mathbf{r}'_1)} \frac{\delta}{i\delta\eta(\mathbf{r}_n)} \dots \frac{\delta}{i\delta\eta(\mathbf{r}_1)}. \end{aligned} \quad (8.25)$$

This expression is simplified when the number of excited levels  $L$  is large ( $L \gg 1$ ). Then all functions  $Q_n$  with  $n \geq 2$  will be small with respect to the parameter  $L^{-n+1}$  and can be neglected. In this limit the thermoequilibrium distribution  $\mathcal{P}_T\{\psi(\mathbf{r}), \psi^*(\mathbf{r})\}$  is equal to

$$\mathcal{P}_T\{\psi(\mathbf{r}), \psi^*(\mathbf{r})\} \approx N \exp\left(-Z \int d^d\mathbf{r} \psi^*(\mathbf{r}) e^{\beta\hat{H}(\mathbf{r})} \psi(\mathbf{r})\right). \quad (8.26)$$

The normalisation factor  $N$  can be determined explicitly in the eigenfunction representation:

$$\psi(\mathbf{r}) = \sum_j a_j \psi_j(\mathbf{r}). \quad (8.27)$$

This gives

$$\begin{aligned} \int \mathcal{P}_T\{\psi(\mathbf{r}), \psi^*(\mathbf{r})\} \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) &= N \int \exp\left(-Z \sum_j e^{\beta E_j} a_j^* a_j\right) \prod_j da_j^* da_j = 1 \end{aligned} \quad (8.28)$$

or

$$N = \prod_j (Z e^{\beta E_j} / 2\pi). \quad (8.29)$$

The averaging of any operator is quite standard:

$$\begin{aligned}
 \langle \hat{B} \rangle_T &= \int \int \psi^*(\mathbf{r}) \hat{B} \psi(\mathbf{r}) \mathcal{P}_T\{\psi(\mathbf{r}), \psi^*(\mathbf{r})\} d^d \mathbf{r} \mathcal{D}\psi(\mathbf{r}) \mathcal{D}\psi^*(\mathbf{r}) \\
 &= \sum_{j,k} \int a_j^* a_k \langle j | \hat{B} | k \rangle \mathcal{P}_T\{a_m, a_m^*\} \prod_m da_m^* da_m \\
 &= Z^{-1} \sum_j e^{-\beta E_j} \langle j | \hat{B} | j \rangle
 \end{aligned} \tag{8.30}$$

and coincides with the general thermodynamic expression (see, e.g., Landau and Lifshitz 1969). Using the terminology of quantised fields the theory may be called 'free' if  $\mathcal{P}_T\{\psi(\mathbf{r}), \psi^*(\mathbf{r})\}$  depends only on a bilinear combination of  $\psi(\mathbf{r})$  and  $\psi^*(\mathbf{r})$ . We see that in the general case the effective field theory is not 'free' and is non-linear due to the effects of exact conservation of normalisation. The other interesting problem consists of the derivation of the distribution (8.24) in the form of a variational principle for effective entropy in the space of  $\psi$  functions (which is also expressed by a continuous integral) subjected to certain additional restrictions due to conservation laws.

The quantum Fokker-Planck equation (8.14) in the space of  $\psi$  functions contains quasidiffusive terms only. For this reason, by analogy with classical theory it is tempting to introduce special additional 'viscous' terms. Their structure is unambiguously determined as in classical theory by the thermoequilibrium distribution. The final result can be compactly written in the form

$$\begin{aligned}
 \frac{\partial \mathcal{P}}{\partial t} &= \hat{L} \mathcal{P} + \frac{1}{2} \int_0^t d\tau \int \int d^d \mathbf{r} d^d \mathbf{r}' \hat{\Gamma}(\mathbf{r}, t; \mathbf{r}', \tau) \\
 &\quad \times (\hat{D}(\mathbf{r}) \mathcal{P}_T \hat{D}(\mathbf{r}') + \hat{D}(\mathbf{r}') \mathcal{P}_T \hat{D}(\mathbf{r})) (\mathcal{P}_T^{-1} \mathcal{P})
 \end{aligned} \tag{8.31}$$

where  $\mathcal{P}_T$  is the thermoequilibrium distribution and the other notation is as in equations (8.1)–(8.4) and (8.15). The addition of such a 'viscosity' violates the superposition principle. We have succeeded in reproducing with this term a number of results from the various semiphenomenological approaches which are usually non-linear and where the superposition principle is not satisfied. The only function of such 'viscous' terms is to ensure evolution to thermoequilibrium. As has been proved in § 4 such an evolution is obtained within linear theory without violation of the superposition principle. Thus, 'viscous' terms in equation (8.31) are really superfluous and equation (8.14) gives the correct solution of the problem.

## 9. Discussion

To summarise, we should note that the Schrödinger equation with Gaussian dynamic fluctuating potentials gives a convenient model of an open quantum system. It is important that this formalism does not violate the superposition principle. An open system coupled with a thermoequilibrium environment will always evolve to a thermodynamic density matrix in accordance with general principles. The evolution at high temperatures can be approximated by the quasiclassical Fokker-Planck equation. The formalism operates with directly observable quantities, e.g., the correlations of fluctuations of fields can be calculated with the well developed methods of Green functions.

The formalism developed gives a natural quantum generalisation of classical Langevin theory and inherits all its advantages and disadvantages. It describes a number of physically interesting situations but the corresponding formalism is not purely exact and exhaustive. As is seen from the example in § 3, the Gaussian approximation is valid when direct emission-absorption processes dominate, while, e.g., the multiphonon processes are neglected. On the other hand, the formalism is not restricted by quasiparticle approximation of environmental fluctuations which may have, e.g., the Ornstein-Uhlenbeck form and, thus, may deviate greatly from quasiparticle behaviour. An analogous theory can be used for description of the dynamics of dilute spin systems in random fluctuating magnetic fields (Chechetkin and Lutovinov 1987) and for describing fluctuating vector potentials.

The model is applicable both for thermoequilibrium and non-equilibrium fluctuations. This last remark concerns artificial sources of stochastic radiation and turbulent media primarily. Thus, this model can be applied to a variety of problems in solid state and plasma physics.

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### Appendix 1

For the convenience of a reader who is interested primarily in quantum mechanical applications we add here some elementary comments on the structure of equation (8.1). For specialists in stochastics we should note that the Stratonovich stochastic rule is used everywhere throughout the following discussion.

Let us consider the one-dimensional stochastic equation

$$\frac{dx}{dt} = f(x)\xi(t) \quad (\text{A1.1})$$

where the Gaussian random force  $\xi(t)$  is defined by the correlators

$$\langle \xi(t) \rangle = 0 \quad (\text{A1.2})$$

$$\langle \xi(t)\xi(t') \rangle = D\delta(t-t') \quad (\text{A1.3})$$

and  $f(x)$  is a given function of  $x$ . After replacement of variables

$$y(x) = \int_{x_0}^x \frac{dx'}{f(x')} \quad (\text{A1.4})$$

one obtains

$$dy/dt = \xi(t). \quad (\text{A1.5})$$

It is well known that the stochastic equivalent of equation (A1.5) is given by the Fokker-Planck equation:

$$\frac{\partial \mathcal{P}(y, t)}{\partial t} = \frac{1}{2} D \frac{\partial^2 \mathcal{P}(y, t)}{\partial y^2}. \quad (\text{A1.6})$$

Using conservation of normalisation:

$$\int \mathcal{P}(x, t) dx = \int \mathcal{P}(y, t) dy \tag{A1.7}$$

and equation (A1.4) this can be rewritten in the form

$$\frac{\partial \mathcal{P}(x, t)}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x} \left[ f(x) \left( \frac{\partial}{\partial x} f(x) \mathcal{P}(x, t) \right) \right]. \tag{A1.8}$$

Thus, we see that for multiplicative noise (A1.1) one should use the generalised operator  $(\hat{\partial}/\partial x)f(x)$  whose action is equivalent to multiplication by the function  $f(x)$  and subsequent differentiation. If a given function  $\varphi(x)$  is added to the right-hand side of (A1.1) then the corresponding ‘drift’ term  $-\partial/\partial x(\varphi(x)\mathcal{P}(x, t))$  must be added to equation (A1.6). Equations (8.1)–(8.3) generalise this simple one-dimensional problem in two aspects. First, they include two additional discrete variables related to  $\text{Re } \psi$  and  $\text{Im } \psi$  (or  $\psi$  and  $\psi^*$ ) and, second, they include the continuous variable  $\mathbf{r}$  related to the various positions in space.

**Appendix 2**

We give in this appendix the derivation of the equality (8.24). It is useful to rewrite the expression for the characteristic functional (8.16) in the form

$$\begin{aligned} &\Phi_T^{(\psi)}\{\lambda(\mathbf{r}), \lambda^*(\mathbf{r})\} \\ &= \exp\left(-\hat{Q}\left\{\frac{\delta}{i\delta\eta(\mathbf{r})}, \frac{\delta}{i\delta\eta^*(\mathbf{r})}\right\}\right) \\ &\quad \times \exp\left(-\int \int d^d\mathbf{r} d^d\mathbf{r}' \lambda^*(\mathbf{r}') \langle \psi^*(\mathbf{r}') \psi(\mathbf{r}) \rangle_T \lambda(\mathbf{r}) \right. \\ &\quad \left. - i \int d^d\mathbf{r} \lambda(\mathbf{r}) \eta(\mathbf{r}) - i \int d^d\mathbf{r} \lambda^*(\mathbf{r}) \eta^*(\mathbf{r})\right) \Big|_{\eta(\mathbf{r})=\eta^*(\mathbf{r})=0} \end{aligned} \tag{A2.1}$$

where the notation is defined by equations (8.18), (8.19) and (8.25). The reciprocal Fourier transform of the functional

$$\begin{aligned} &F\{\lambda(\mathbf{r}), \lambda^*(\mathbf{r}); \eta(\mathbf{r}), \eta^*(\mathbf{r})\} \\ &\equiv \exp\left(-\int \int d^d\mathbf{r} d^d\mathbf{r}' \lambda^*(\mathbf{r}') \langle \psi^*(\mathbf{r}') \psi(\mathbf{r}) \rangle_T \lambda(\mathbf{r}) \right. \\ &\quad \left. - i \int d^d\mathbf{r} (\lambda(\mathbf{r}) \eta(\mathbf{r}) + \lambda^*(\mathbf{r}) \eta^*(\mathbf{r}))\right) \end{aligned} \tag{A2.2}$$

is given by

$$\int F\{\lambda(\mathbf{r}), \lambda^*(\mathbf{r}); \eta(\mathbf{r}), \eta^*(\mathbf{r})\} \exp\left(-i \int d^d\mathbf{r} (\lambda(\mathbf{r}) \psi(\mathbf{r}) + \lambda^*(\mathbf{r}) \psi^*(\mathbf{r}))\right) \mathcal{D}\lambda(\mathbf{r}) \mathcal{D}\lambda^*(\mathbf{r}) \tag{A2.3}$$

and is equal to

$$N \exp\left(-\int \int d^d \mathbf{r}' d^d \mathbf{r}'' (\psi^*(\mathbf{r}') + \eta^*(\mathbf{r}'')) G(\mathbf{r}', \mathbf{r}) (\psi(\mathbf{r}) + \eta(\mathbf{r}))\right) \quad (\text{A2.4})$$

where

$$\int d^d \mathbf{r}'' G(\mathbf{r}, \mathbf{r}'') \langle \psi^*(\mathbf{r}'') \psi(\mathbf{r}') \rangle_T = \delta(\mathbf{r} - \mathbf{r}') \quad (\text{A2.5})$$

and  $N$  is some constant. Using the direct definition of the density matrix (4.11) it is easy to see that

$$G(\mathbf{r}', \mathbf{r}) = Z \delta(\mathbf{r} - \mathbf{r}') \exp(\beta \hat{H}(\mathbf{r})). \quad (\text{A2.6})$$

Collecting equalities (A2.1)–(A2.6) one reproduces equation (8.24).

*Note added in proof.* The more careful treatment (cf Kus and Wodkiewicz 1982) shows that Wick's theorem is fulfilled only approximately in the interaction representation and the exact averaged expression must be written as

$$\begin{aligned} & \left\langle \hat{T} \exp\left(-\frac{i}{\hbar} \int_0^t d\tau \hat{V}^{(int)}(\mathbf{r}, \tau)\right) \right\rangle \\ &= \hat{T} \exp\left(-\frac{1}{2\hbar^2} \int_0^t d\tau_1 \int_0^t d\tau_2 \langle \hat{V}^{(int)}(\mathbf{r}, \tau_1) \hat{V}^{(int)}(\mathbf{r}, \tau_2) \rangle\right). \end{aligned}$$

If, however, the inequality

$$\frac{\tau_{\text{corr}}}{\hbar m} \int_0^\infty d\tau \tau \left\langle \frac{\partial V(\mathbf{r}, \tau)}{\partial \mathbf{r}} \frac{\partial V(\mathbf{r}, 0)}{\partial \mathbf{r}} \right\rangle \ll 1$$

is satisfied then

$$\begin{aligned} & \hat{T} \exp\left(-\frac{1}{2\hbar^2} \int_0^t d\tau_1 \int_0^t d\tau_2 \langle \hat{V}(\mathbf{r}, \tau_1) \hat{V}(\mathbf{r}, \tau_2) \rangle\right) \\ & \approx \exp\left(-\frac{1}{2\hbar^2} \hat{T} \int_0^t d\tau_1 \int_0^t d\tau_2 \langle \hat{V}(\mathbf{r}, \tau_1) \hat{V}(\mathbf{r}, \tau_2) \rangle\right). \end{aligned}$$

and all other considerations remain unchanged.

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